ML-PMHT Threshold Determination for False Track Probability Using Extreme-Value Analysis

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***Abstract*—The Maximum Likelihood Probabilistic Multi-Hypothesis Tracker (ML-PMHT) can be used as a powerful multisensor, low-observable, multitarget tracker. It is a non-Bayesian algorithm that uses a generalized likelihood ratio (LR) test to differentiate between clutter and target tracks. Prior to this work, the detection threshold used for target track acceptance was determined either through trial and error or with lengthy Monte-Carlo simulations. We present a new method for determining this threshold by assuming that the clutter is uniformly distributed in the search space, and then treating the log-likelihood ratio (LLR) as a random variable transformation. In this manner, we can obtain an expression for the PDF of the likelihood function caused by clutter. We then use extreme value theory to obtain an expression for the PDF of the peak point of the LLR surface due to clutter. From this peak PDF, we can then calculate a threshold based on some desired (small) false track acceptance probability.**

Keywords: ML-PDA, ML-PMHT, multistatic, bistatic, tracking, extreme value, thresholds for track acceptance

I. Introduction

The Maximum Likelihood Probabilistic Multi-Hypothesis Tracker (ML-PMHT) is a powerful multisensor, low-observable (i.e. received target SNR less than 10 dB even after signal processing) multitarget tracker. It is a variant of the Maximum Likelihood Probabilistic Data Association (ML-PDA) tracker, which was first developed in [8] and was subsequently expanded in [4] and [9]. The idea behind ML-PMHT originated with [1], [16], [17] and [18], with more recent work done in [13] and [15]. These last two works highlight some advantages for ML-PMHT over ML-PDA — ML-PMHT can be easily implemented in true multitarget form, and it has a relatively straightforward and fast method for estimating the covariance of the state estimate. The LLR (log-likelihood ratio) for ML-PMHT is also simpler to express than the LLR for ML-PDA, which makes the random variable transformations that will be done in this work tractable. For these reasons, working with the ML-PMHT LLR is the focus of this paper.

Previous work in [2] presented a method for determining a track acceptance threshold for ML-PDA; the same method can be applied to ML-PMHT. Basically, for a certain set of problem parameters such as search volume, clutter density, probability of detection, signal-to-noise ratio, etc., the ML-PMHT LLR can be simulated and then maximized; if this is done repeatedly with Monte Carlo trials the empirical PDF (probability density function) of the maximum point in the LLR due to clutter can be obtained. The problem with this approach is that it is an extremely time-consuming process, taking anywhere from hours to days, and if any of the parameters change, the thresholds must be recomputed. Often, this threshold determination method takes long enough that in past works [6], [12], a combination of this method and trial and error was used to find an appropriate tracking threshold for ML-PMHT implementations. The work in [2] did also develop a "real-time" method for track-validation for ML-PDA, but its accuracy was not as good as the "off-line" approach described above. We seek the accuracy of the off-line method in real-time speed.

To do this, we take a novel approach to the problem. Instead of having a fixed set of measurements Z and finding the state parameter vector $\mathbf{x}$ that maximizes the ML-PMHT LLR for the received measurements, we assume we have some arbitrary parameter $\mathbf{x}$, and all the measurements Z are random variables that are uniformly distributed when there is no target in the search space (which, as will soon be discussed, can be anywhere from one- to three-dimensional). The ML-PMHT LLR is simply treated as a transformation of the clutter (uniformly distributed) random variables. This transformation produces an expression that represents the PDF of the ML-PMHT LLR in the absence of a target. We then employ extreme value theory [3], [5], [7] to develop a PDF for the maximum value of the LLR. With the PDF describing the maximum value in the LLR due to clutter, we can then employ the Neyman-Pearson lemma [11] to obtain a threshold that will give us a certain false track acceptance probability.

II. Extreme-Value Clutter PDFs

Extreme-value (EV), namely, Gumbel PDFs caused by clutter are calculated for four cases: one-dimensional measurements (bearing-only or time-delay-only), two-dimensional measurements (bearing and time-delay), three dimensional measurements (bearing, time-delay, and Doppler), or two-

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dimensional measurements with amplitude. First consider the ML-PMHT log-likelihood ratio (LLR), which can be written as [12]

$$\Lambda(\mathbf{x}, Z) = \sum_{i=1}^{N_w} \sum_{j=1}^{m_i} \ln \left\{ 1 + \frac{\pi_1}{\pi_0} V p[\mathbf{z}_j(i)|\mathbf{x}] \right\} \quad (1)$$

where N_w is the number of scans in a batch, Z is the entire set of measurements in a batch, m_i is the number of measurements in the i^{th} scan, $\mathbf{z}_j(i)$ is the j^{th} measurement in the i^{th} scan, π_1 is the prior probability that a given measurement is from the target, π_0 is the prior probability that a given measurement is from clutter, V is the measurement search volume, $p[\mathbf{z}_j(i)|\mathbf{x}]$ is a target-centered Gaussian, and $\mathbf{x}$ is the target parameter vector. Now consider the LLR for just a single measurement from (1), which is written in a slightly modified form

$$\Lambda_1(\mathbf{z}) = \ln \left\{ 1 + K_d \exp \left[-\frac{1}{2} \sum_{l=1}^{d} \frac{(z_l - \mu_l)^2}{\sigma_l^2} \right] \right\} \quad (2)$$

where z_l is the l^{th} component of $\mathbf{z}$ and we define

$$K_d = \frac{\pi_1}{\pi_0} \frac{V}{\sqrt{|2\pi \mathbf{R}_d|}} \qquad d \in \{1, 2, 3\} \quad (3)$$

Here d is the dimensionality of the measurement, and $\mathbf{R}_d$ is the measurement covariance matrix. Now, simply treat (2) as the transformation of a random variable

$$w \triangleq \Lambda_1(\mathbf{z}) \quad (4)$$

If the PDF of a scalar z is given by $p_z(z)$, the PDF of w is given by

$$p_w(w) = p_z[\Lambda_1^{-1}(w)] \left| \frac{dz}{dw} \right| \quad (5)$$

For a one-dimensional uniformly distributed (clutter) measurement, the transformed PDF is

$$p_w(w) = \frac{2}{V} \frac{\sigma_1 \exp(w)}{\sqrt{2 \ln \frac{1-\exp(w)}{K_1}}} \frac{1}{\exp(w) - 1} \qquad 0 \le w \le \ln(1 + K_1) \quad (6)$$

For two-dimensional measurements, the transformed PDF is [14]

$$p_{w_2}(w_2) = \begin{cases} C\delta(w_2) & w_2 = 0 \\ 2\pi \frac{\sigma_1 \sigma_2}{V_1 V_2} \frac{\exp(w_2)}{\exp(w_2) - 1} & 0 < w_2 \le \ln(1 + K_2) \end{cases} \quad (7)$$

where δ is a Dirac delta function and C is a normalization constant that is scaled so that the entire PDF integrates to 1. For three dimensions, the transformed PDF is [14]

$$p_{w_3}(w_3) = \begin{cases} C\delta(w_3) & w_3 = 0 \\ 4\pi\sqrt{2} \frac{\sigma_1 \sigma_2 \sigma_3}{V_1 V_2 V_3} \frac{\exp(w_3)}{\exp(w_3) - 1} \sqrt{\ln\left(\frac{K_3}{\exp(w_3) - 1}\right)} & 0 < w_3 \le \ln(1 + K_3) \end{cases} \quad (8)$$

Finally, a closed-form solution can also be obtained for the 2-D case with amplitude information, where both the target and

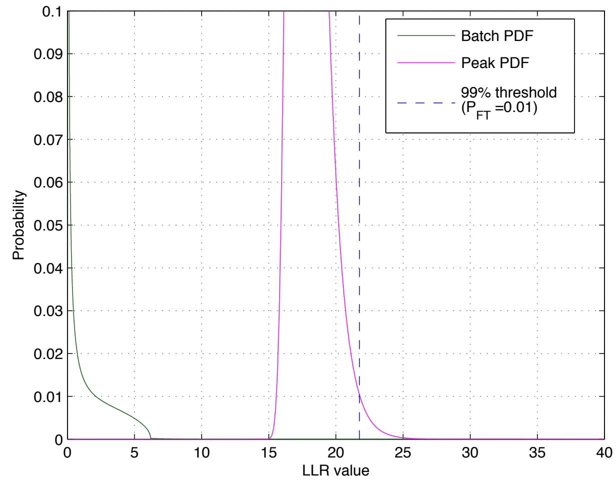

Figure 1. Three-dimensional ML-PMHT PDFs

clutter amplitude have a Rayleigh distribution. The resultant PDF in this case is [14]

$$p_{w_{2a}}(w_{2a}) = \begin{cases} C\delta(w_{2a}) & w_{2a} = 0 \\ \frac{2\pi\sigma_1\sigma_2}{V_1 V_2} \left[1 - e^{\tau} \left(\frac{e^{w_{2a}} - 1}{K'} \right)^{\frac{1}{K_\sigma}} \right] \frac{\exp(w_{2a})}{\exp(w_{2a}) - 1} & 0 < w_{2a} \le \ln\left(1 + K_2' e^{-K_\sigma \tau}\right) \end{cases} \quad (9)$$

Here, τ is the detector threshold in units of intensity, σ^2 is the expected target power, and

$$K_2' = \frac{K_2}{\sigma^2} e^{K_\sigma \tau} \quad (10)$$

and

$$K_\sigma = \frac{1 - \sigma^2}{\sigma^2} \quad (11)$$

At this point, we have the PDF for a single transformed clutter measurement – now we want the PDF for a batch of measurements, as represented by (1). This can done simply by adding the N IID (independent identically distributed) single-measurement RVs (where N is the total number of measurements in the batch — i.e. $N = \sum_{i=1}^{N_w} m_i$), each of which has the common PDF (dependent on the measurement dimensionality) given by (6), (7), (8), or (9). Such a PDF can be calculated in two ways — either by directly convolving the relevant single-measurement PDF with itself $N - 1$ times, or calculating its characteristic function, raising it to the power N, and then transforming this product back to a PDF. An example result for the batch and peak PDFs (to be discussed below) for the 3-D case is shown in Figure 1.

At this point, we have batch PDFs; they represent the possible values of the *entire* ML-PMHT LLR surface. However, we wish to describe the PDF of the *peak* point of the clutter-induced LLR. This is done by invoking extreme-value theory. If the batch PDF is sampled M times, and the samples are ordered, then the PDF of the maximum sample can be equated to the peak clutter PDF. We choose M such that if we were sampling the ML-PMHT LLR on a grid, the maximum sample

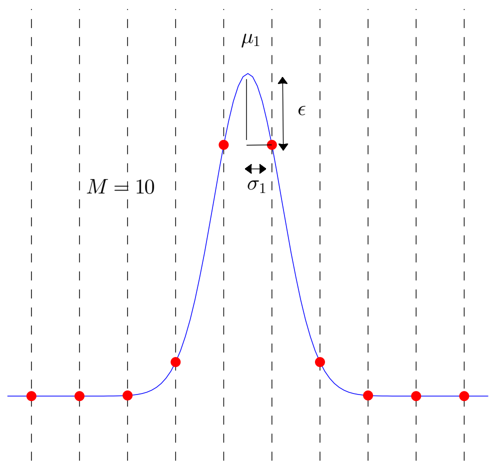

Figure 2. Relationship between number of samples M and distance ϵ from LLR peak. The blue curve is a representative global peak in the ML-PMHT LLR.

would be as close to the "true" peak caused by clutter as we could get by using some iterative optimization technique. Figure 2 illustrates this. From this, the number of samples M required to get within ϵ of the true peak (the same accuracy we would get with optimization) can be shown to be [14]

$$M = \frac{1}{2}\sqrt{\frac{m_{ave}V}{\sigma_1\epsilon}\frac{K_1}{K_1+1}} + 1 \tag{12}$$

Here, σ_1 is the measurement noise standard deviation (in one dimension), and m_{ave} is the average number of measurements in a scan. Overall, the required number of measurements must be calculated for each measurement dimension; the total number of required measurements M_{tot} is then the product of the required number of measurements from each dimension.

At this point we have the PDF for the entire ML-PMHT LLR surface caused by clutter, and the number of samples M_{tot} needed to obtain the accuracy ϵ that would be obtained by a properly implemented numerical optimization routine. If the PDF for the entire ML-PMHT LLR surface (caused just by clutter) is sampled M_{tot} times, and the samples are ordered, the PDF of the maximum sample can be used to represent the PDF of the peak in the LLR that would be obtained through optimization.

Previous work from [2] showed that this peak PDF is a Gumbel distribution. This distribution has the form [7]

$$f(x) = \frac{1}{\beta}\exp\left[-\left(\frac{x-\nu}{\beta}\right) - \exp\left(-\frac{x-\nu}{\beta}\right)\right] \tag{13}$$

If $F^{-1}(x)$ is the inverse CDF of the underlying distribution that is being sampled, then the parameters ν and β in (13) can be obtained as [3]

$$\nu = F^{-1}\left(1 - \frac{1}{M_{tot}}\right) \tag{14}$$

and

$$\beta = F^{-1}\left(1 - \frac{1}{eM_{tot}}\right) - \nu \tag{15}$$

One possible concern is that the above extreme value theory applies to a sequence of IID random variables; if we actually sample the ML-PMHT LLR at a close enough grid spacing, there will be dependence between some samples. This is because samples that are relatively close together will be influenced (i.e. the LLR will take on value) from the same measurement, so intuitively, these samples will be dependent. However, other work in extreme value theory [10] shows that the PDF of the maximum sample from an identical yet dependent sequence will asymptotically converge to the IID case. For the ML-PMHT, (12) will produce values of M_{tot} that are large enough (typically $10^7 \leq M_{tot} \leq 10^{10}$) so that it is reasonable to assume this convergence has occurred.

In order to obtain the Gumbel distribution parameters, it is necessary to obtain the CDF of the distribution that represents a batch of measurements for the ML-PMHT LLR. There is no convenient closed-form expression for this CDF that can easily be inverted. The closest available CDF will be a numerical integration of the PDF that represents the batch ML-PMHT LLR. In theory it is simple to numerically invert the CDF of the batch and find the constants ν and β from (14) and (15). However, the number M_{tot} is typically large, so $0.99 < 1 - 1/M_{tot},\ 1 - 1/eM_{tot} < 1$. Trying to numerically invert the CDF at the extreme right side can lead to numerical calculation inaccuracies. It is more accurate to approximate the right-hand side of the CDF with the easily invertible closed-form expression. To this end, the CDF $F(x)$ is approximated with

$$F_{approx}(x) = 1 - \exp\left[-k(x-m)^l\right] \tag{16}$$

The parameters $k, m,$ and l are fit to the numerical CDF in the region $F(x) > 0.95$ using Matlab's Optimization Toolbox. This expression is easily invertible; letting $F_{approx}(x) = \phi$ and solving for x produces

$$x = m + \left[-\frac{1}{k}\ln(1-\phi)\right]^{\frac{1}{l}} \tag{17}$$

At this point, combining (17), (14), and (15) gives final expressions for the parameters of the Gumbel distribution

$$\nu = m + \left[\frac{1}{k}\ln(M_{tot})\right]^{\frac{1}{l}} \tag{18}$$

and

$$\beta = \left[\frac{1}{k} + \frac{1}{k}\ln(M_{tot})\right]^{\frac{1}{l}} - \left[\frac{1}{k}\ln(M_{tot})\right]^{\frac{1}{l}} \tag{19}$$

With this, it is possible to write the Gumbel distribution that represents the peak in the ML-PMHT LLR surface caused by clutter.

III. Threshold Determination

At this point, the extreme value distribution (peak PDF) from clutter is used to generate tracking thresholds. These results are compared to thresholds developed via the (much slower) method described in [2] — i.e. repeatedly simulating and optimizing a (clutter-only) ML-PMHT LLR surface.

Table I
VALUE USED FOR THRESHOLD DETERMINATION

1-Dimensional Parameters	
Angular volume	180°
Angular error	2°
Ave meas per scan	10
Number of scans	60
π_1	0.05
π_0	0.95
2-D and 3-D Parameters	
Angular volume	360°
Angular error	5°
Time delay volume	60 sec
Time delay error	0.1 sec
Range-rate volume	30 units per sec
Range-rate error	0.5 units per sec
Amplitude threshold	7 dB
Expected target amplitude	10 dB
Ave meas per scan	9.8
Number of scans	11
π_1	0.15
π_0	0.85

Table II
COMPARISON OF MODEL VS. EMPIRICAL TRACKING THRESHOLDS (DIMENSIONLESS UNITS FOR LLR)

	Model	Empirical	Empirical 95% Confidence Interval
1-D	41.8	42.0	[41.9, 42.2]
2-D	27.0	27.3	[26.7, 27.9]
3-D	21.8	21.9	[21.5, 22.4]
2-D amp	23.5	23.0	[22.5, 23.6]

For the Gumbel distribution, the CDF (and a threshold) is easily calculable once the parameters for the PDF are known. For a Gumbel distribution (13), the CDF is given by

$$F(x) = \exp\left(-e^{-\frac{x-\nu}{\beta}}\right) \tag{20}$$

Then, for a desired probability of false track acceptance L, the threshold κ is found by inverting the CDF

$$\kappa = \nu - \beta \ln\left(\ln \frac{1}{1-L}\right) \tag{21}$$

The problem parameters that were used to calculate "model" thresholds are listed in Table I. Next, results from the model threshold determination and actual Monte Carlo testing are shown in Table II. (The value of L used for both the model and empirical results was 0.01.) The model-based threshold values are very close to the actual empirical threshold values obtained from Monte Carlo testing. It was far faster to calculate the model-based values — each needed only on the order of several seconds. In contrast, the actual values obtained by simulating and optimizing multiple clutter surfaces were much more time-consuming to calculate — for 5000 Monte Carlo runs, it took on the order of 10 hours to obtain each value.

IV. CONCLUSIONS

We have presented a novel method for determining the PDFs describing the maximum points in the ML-PMHT LLR caused by clutter using random variable transformations. The EV distribution describing the peak point (global maximum) in the LLR due to clutter can be used to quickly and easily determine a track acceptance threshold; previously, this had to be done either through trial-and-error or with very time consuming optimizations of Monte Carlo simulations. Thresholds obtained in this manner match up extremely well with thresholds obtained by (the much more time consuming) Monte-Carlo simulation and optimization of ML-PMHT LLR surfaces. The ability to rapidly and accurately determine tracking thresholds adds significant capability to the ML-PMHT multistatic active tracking framework and other applications.

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