Source Number Detection with Nested Arrays and ULAs Using Jackknifing

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Abstract—We consider the problem of source number detection based on uniform linear arrays (ULAs) and the recently proposed nested arrays. A ULA with \(N\) sensors can detect at most \(N-1\) sources, whereas a nested array provides \(O(N^2)\) degrees of freedom with \(O(N)\) sensors, enabling us to detect \(K\) sources with \(N<K\) sensors. In order to make full use of the available limited valuable data, we propose a novel strategy, which is inspired by the jackknifing resampling method. Exploiting numerous iterations of subsets of the whole data set, this strategy helps the existing detection methods achieve great improvements. Numerical simulations demonstrate the advantage of our strategy, both for ULAs and nested arrays.

Index Terms—Source number detection, nested array, jackknifing, uniform linear array

I. INTRODUCTION

Source number detection is a prerequisite for direction of arrival (DOA) estimation. The use of a ULA for source number detection has received a considerable amount of attention in the last three decades [1]-[8]. Various methods have been proposed according to different mathematical criteria. The most commonly used techniques are based on information theoretic criteria, such as the Akaike information criterion (AIC) [3], the Kullback-Leibler information criterion (KIC) [4], and Rissanen’s minimum description length (MDL) [5] principle. These methods conduct detection by combining eigenvalue decomposition, the maximum likelihood function, and some penalty functions. Another eigenvalue-based method, called second order statistic of eigenvalues (SORTE) [6], is based on a gap measure of the eigenvalues. A predicted eigenthreshold (ET) approach was proposed by Chen [7], which detects the number of sources by setting an upper bound on the eigenvalues and then implementing a hypothesis testing procedure. All the aforementioned methods are based on eigenvalues of the sample covariance matrix. Eigenvectors can also be used for the determination of sources. Jiang and Ingram [8] proposed an eigenvector-based method by exploiting the property of the variance of the rotational submatrix (VTRS).

All the existing detection methods exploit the available data by calculating the whole sample covariance matrix. However, this fails to make full use of the available limited information. Jackknifing [9] is a resampling strategy used to estimate sample statistics by using subsets of the available data. Jackknifing [10] helps fully exploit the received data to improve the detection performance.

A ULA with \(N\) sensors can detect at most \(N-1\) sources. A systematic approach to achieve \(O(N^2)\) degrees of freedom using \(O(N)\) sensors based on a nested array was recently proposed in [11], where DOA estimation and beamforming were studied. This nested array can detect more sources than the number of sensors [12].

In this paper, we propose a novel strategy, jackknifing, to detect the number of sources. Both the ULA and the nested-array are considered. Simulations are provided to demonstrate the advantage of our strategy.

II. SIGNAL MODEL

We assume there is a nonuniform linear nested array with \(N\) sensors, including two concatenated uniform linear arrays (ULA). Suppose the inner ULA has \(N_1\) sensors with spacing \(d_I\) and the outer ULA has \(N_2\) sensors with spacing \(d_O = (N_1 + 1)d_I\). We assume \(K\) narrowband sources are in the surveillance region, impinging on this linear array from directions \(\{\theta_k, k = 1, \ldots, K\}\). We can obtain the received signal as

\[
y(t) = Ax(t) + e(t),
\]

where \(y(t) = [y_1(t), y_2(t), \ldots, y_N(t)]^T\) is the \(N\) received signal at time \(t\). Let \(a(\theta_k)\) be the \(N \times 1\) steering vector with the \(i\)th element \(e_j(2\pi/\lambda)d_isin\theta_k\). \(d_i\) is the position of \(i\)th sensor, which is an integer multiple of the basic spacing \(d_I\) or \(d_O\). \(\lambda\) denotes the carrier wavelength. Then the manifold matrix can be expressed as \(A = [a(\theta_1), a(\theta_2), \ldots, a(\theta_K)]\), \(x(t) = [x_1(t), x_2(t), \ldots, x_K(t)]^T\) is the source vector. We suppose the source signals follow Gaussian distributions, \(x_k \sim N(0, \sigma_k^2)\), and they are all independent of each other. The noise signal \(e(t) = [e_1(t), e_2(t), \ldots, e_N(t)]^T\) is assumed to be white Gaussian, and uncorrelated with the sources.

Suppose we have \(T\) measurements. Stacking all the measurements together, we rewrite (1) as

\[
Y = AX + E,
\]

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where

- \( Y = [y(1), y(2), \ldots, y(T)], \) an \( N \times T \) matrix,
- \( X = [x(1), x(2), \ldots, x(T)], \) an \( K \times T \) matrix,
- \( E = [e(1), e(t), \ldots, e(T)], \) an \( N \times T \) matrix.

Based on our assumption, the source autocorrelation matrix \( R_x \) is diagonal: \( R_x = \text{diag}(\sigma^2_1, \sigma^2_2, \ldots, \sigma^2_K). \) Then the autocorrelation matrix of the received signal is

\[
R_y = AR_xA^H + \sigma^2 I,
\]

where \( \sigma^2 \) is the noise power, and \( I \) is the identity matrix. Vectorizing \( R_y \), we get:

\[
v = (A^* \otimes A)p + \sigma^2 e_1,
\]

where \( p = [\sigma^2_1, \sigma^2_2, \ldots, \sigma^2_K]^T, \) and \( e_1 = [e_1^T, e_2^T, \ldots, e_K^T]^T, \) with \( e_i \) being a vector of all zeros except a 1 at the \( i \)th position. We can view vector \( v \) in (3) as some new longer received signals with the new manifold matrix \( A^* \otimes A, \) and the new source signals \( p. \) \( \otimes \) denotes conjugation without transpose, and \( \otimes \) denotes the Khatri-Rao product.

### III. Source Detection

We will use the nested array mentioned above to conduct detection of more sources than the number of sensors. First, we will briefly introduce spatial smoothing [11], which is used to exploit the increased degrees of freedom. Then, we will list four source detection methods, and propose a new strategy, which applies jackknifing to improve the detection performance.

#### A. Spatial Smoothing

In order to exploit the increased degrees of freedom provided by the co-array, we need to apply spatial smoothing. We remove the repeated rows from \( A^* \otimes A \) and also sort them so that the \( i \)th row corresponds to the sensor location \( (-N^2/4 - N/2 + i) d_i \) in the difference co-array of the 2-level nested array, giving a new vector: \( \bar{v} = \bar{A}p + \sigma^2 \bar{e}, \) where \( \bar{e} \in \mathbb{R}^{((N^2-2)/2+N) \times 1} \) is a vector of all zeros except a 1 at the center position.

The difference co-array of this 2-level nested array has sensors located at \((-N^2/4-\frac{N}{2}+1)d_1, \ldots, -d_1, 0, d_1, \ldots, (N^2/4+N/2-1)d_1).\)

We now divide these \( N^2/2 + N - 1 \) sensors into \( N^2/4 + N/2 \) overlapping subarrays, each with \( N^2/4 + N/2 \) elements, where the \( i \)th subarray has sensors located at \( \{-i+1+n\}d_i, \ n = 0, 1, \ldots, \frac{N^2}{4} + \frac{N}{2}-1 \). The \( i \)th subarray corresponds to the \((-N^2/4 + N/2 - 1 + i)\)th to \((N^2 - N - i)\)th rows of \( \bar{v} \), denoted as \( \bar{v}_i = \bar{A}_i p + \sigma^2 \bar{e}_i. \)

We can check that \( \bar{v}_i = \bar{A}_i \Phi^{-1} p + \sigma^2 \bar{e}_i, \) where \( \Phi = \text{diag}(e^{-j(2\pi/\lambda) \sin \theta_{i1}}, e^{-j(2\pi/\lambda) \sin \theta_{i2}}, \ldots, e^{-j(2\pi/\lambda) \sin \theta_{iK}}). \)

Viewing \( \bar{v}_i \) as a newly received vector, we can get the equivalent covariance matrix \( R_i = \bar{v}_i \bar{v}_i^H. \) Taking the average of \( R_i \), we get

\[
R_{\text{ave}} = \frac{1}{\left(\frac{N^2}{4} + \frac{N}{2}\right)} \sum_{i=1}^{N^2/4+N/2} R_i.
\]

The spatially smoothed matrix \( R_{\text{ave}} \) enables us to perform detection of \( O(N^2) \) sources with \( N \) sensors.

#### B. Source Detection Using Jackknifing

As mentioned in the introduction, the sample covariance matrix \( R_y \) is a key element for source detection. Considering a uniform linear array with \( N \) sensors, we do eigenvalue decomposition: \( \text{EVD}(R_y) = U \Lambda U^T, \) where \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \) are the eigenvalues and \( U = [u_1, u_2, \ldots, u_N] \) is the corresponding eigenvector matrix. Suppose the eigenvalues are sorted decreasingly:

\[
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_K > \lambda_{K+1} = \ldots = \lambda_N.
\]

We now briefly introduce four detection methods.

- **SORTE**
  A gap measure is defined:

\[
\text{SORTE}(k) = \begin{cases} \frac{\text{var}(\{\nabla \lambda_i\}_{i=k+1}^{N-1})}{\text{var}(\{\nabla \lambda_i\}_{i=k}^{N-1})} & \text{if } \var(\{\nabla \lambda_i\}_{i=k}^{N-1}) \neq 0 \\ +\infty & \text{if } \var(\{\nabla \lambda_i\}_{i=k}^{N-1}) = 0 \end{cases}
\]

where \( k = 1, \ldots, N-2, \) \( \nabla \lambda_i = \lambda_i - \lambda_{i+1} \) and

\[
\var(\{\nabla \lambda_i\}_{i=k}^{N-1}) = \frac{1}{N-k} \sum_{i=k}^{N-1} (\nabla \lambda_i)^2 + \frac{1}{N-k} \sum_{j=k}^{N-1} (\nabla \lambda_j)^2.
\]

Then the source number is \( K = \arg\min_k \text{SORTE}(k). \)

- **VTRS**
  Suppose \( E_s \) is the combined signal eigenvectors of \( R_y, \) and \( E_x \) and \( E_y \) are the first \( N-1 \) rows and last \( N-1 \) rows of \( E_s \) respectively. Solving \( E_y = E_x \Phi \) based on the least square criterion, we get matrix \( \Phi. \) Define \( \Delta_k = \{\Phi(i,j)\}_{(N-k-1) \times k}, \) \( i = k+1, \ldots, N-1, \) and \( j = 1, \ldots, k. \)

Then the source number is

\[
K = \arg\min_k \frac{\|\Delta_k\|^2}{(N-k-1)K}, \ k = 1, \ldots, N-2,
\]

where \( \| \cdot \| \) is the Frobenius norm.

- **ET**
  Define the eigen-threshold

\[
\bar{\lambda}_{N-m} = \left[\frac{1}{1-t(T(m+1))^{-1/2}} - m\right]_{N-m+1},
\]

where \( t \) is a pre-set parameter, and

\[
l_i = \frac{1}{N-i+1} \sum_{j=1}^{N} \lambda_j, \ i = k+1, \ldots, N.
\]

Based on this, we keep testing the binary hypothesis: \( H_0: K < N - m \) and \( H_1: K = N - m. \) Accept \( H_1 \) or \( H_0 \) according to \( \lambda_{N-m} \leq h_0 \bar{\lambda}_{N-m}. \) If \( H_0 \) is accepted, then we set \( m = m + 1, \) and continue. Otherwise, if \( H_1 \) is accepted, stop testing, and assign \( K = N - m. \)

- **AIC**
  Define

\[
L(k) = \frac{T}{2} \log \left( \prod_{i=k+1}^{N} \frac{1}{\lambda_i^{1/(N-k)}} \right) \sum_{i=k+1}^{N} \lambda_i^{1/(N-k)}.
\]
and \( P(k) = 1 + Nk - 1/2k(k - 1) \). Then the source number is determined as

\[ K = \arg \min_k -2L(k) + 2P(k). \]

All the existing methods are based on the eigenvalues or eigenvectors of \( R_y \). However, the received data can tell us more.

Researchers have been using all the measurements as a whole to get the sample covariance, then studying further based on this covariance matrix. What we will try to do here is make full use of the received data, achieving more accurate detection. Jackknifing is an effective strategy used in statistical area to estimate sample statistics. The idea is to use subsets of available data to improve performance. We propose a new detection strategy based on the idea of jackknifing. Our basic belief is that a large proportion of the available data approximately contains the same amount of information as the whole available data set does.

Suppose we have \( T \) measurements in total:

\[ Y = [y(1), y(2), \ldots, y(T)]. \]

First, we take a subset \( Y_J \) of size \( T_J \) from the \( T \) measurements matrix \( Y \):

\[ Y_J = [y(1), y(2), \ldots, y(T_J)]. \]

Then we do eigenvalue decomposition for \( R_{Y_J} \):

\[ \text{EVD}(R_{Y_J}) = U_J \Lambda_J U_J^H, \]

where \( \Lambda_J = \text{diag}(\lambda_1, \ldots, \lambda_{T_J}) \). Using \( \Lambda_J \) and \( U_J \), we conduct source detection using the existing methods. Suppose we obtain the source number \( \hat{K} \). We continue the above two procedures for \( Z \) iterations, obtaining \( Z \) estimated source numbers, \( \hat{K}_z, z = 1, \ldots, Z \). Before making the decision of the final source number, we need one more step, counting the occurrence of each estimated number, denoted as \( Z_1, Z_2, \ldots, Z_{N-1} \), with summation \( Z \). The final source number is chosen as the one that occurs most frequently:

\[ \hat{K} = \arg \max_k Z_k. \quad (6) \]

The algorithm is shown in Table I.

### Remarks:

- Theoretically, when the detection accuracy is higher than 50%, the detection performance with jackknifing will definitely be improved. Please see the detailed proof in our paper [10]. With over 50% accuracy, the correct number detected should occur more frequently than other numbers do. This assumption is based on the condition that the jackknifing sample subset contains enough information to guarantee over 50% accuracy. Otherwise, the jackknifing will lose its power.
- For methods that are sensitive to sample number, we need to increase the sample number to guarantee the efficiency of jackknifing. For example, one method performs well with the whole \( T \) samples. However, when applying jackknifing, we just use \( rT \) samples, in which case this method may achieve accuracy lower than 50%. Consequently, jackknifing will fail. Alternatively, we can adjust the value of \( r \) to get the accuracy.
- When there is a low signal-to-noise ratio (SNR), namely a high noise level, the detection methods may fail to detect the source number correctly, with accuracy lower than 50%. This will cause jackknifing to perform badly, as discussed in the first remark.

### IV. Numerical Examples

In this section, we use numerical examples to show the superiority of our proposed strategy for source detection for both a ULA and a nested array.

#### A. Uniform Linear Array

We consider a uniform linear array with \( N = 8 \) sensors with equal spacing \( d = \lambda/2 \). Suppose we have \( K = 3 \) sources in the scene, with the same powers \( \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 9 \) and impinging from directions \( \theta = [-60^\circ, 0^\circ, 30^\circ] \). We choose a jackknifing iteration number \( Z = 20 \), the percentage of \( r = 0.85 \), and the Monte Carlo simulation number \( T = 1500 \). Fig. 1 shows the results of the aforementioned four different methods: SORTIE, VTRS, ET, and AIC, with and without jackknifing. It describes the detection accuracy with respect to the SNR. We take the SNR as \( 10\log_{10} \frac{E[x_t^2]}{E[\xi_t^2]} \), and the detection accuracy as \( F_{\hat{K}}/F \), where \( F \) is the trial number, and \( F_{\hat{K}} \) is the number of times that \( \hat{K} \) is detected. In this example, we use \( F = 1000 \) trials.
We can see that both SORTE and VTRS with jackknifing outperform their counterparts without jackknifing. The blue-star line represents the detection accuracy with jackknifing, and the red-circle line without jackknifing. The vertical axis represents the percentage we choose when doing jackknifing. We will also consider applying this strategy to a co-prime array.

B. Nested Array

We consider a 2-level nested array with \( N = 6 \) sensors, both inner and outer ULAs having 3 sensors, with spacing \( d_I = \lambda/2 \) and \( d_O = 4d_I \). Suppose there are 8 sources, with equal power and impinging from directions \( \theta = [-60^\circ, -45^\circ, -15^\circ, 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ] \). It is impossible for us to use a 6-ULAs to detect 8 sources. However, the spatial matrix \( R_{\text{ave}} \) in (4) helps a nested array obtain this goal. We choose jackknifing iteration number \( Z = 20 \), and the percentage \( r = 0.85 \). We conduct \( T = 2000 \) Monte Carlo simulations, and use \( F = 100 \) trials. From Fig. 1, we can see SORTE and VTRS perform better, thus we consider only these two methods for the nested array.

Fig. 2 shows the performance of SORTE and VTRS, with and without jackknifing. We can see that at high SNR both methods can detect the source number correctly with high probability. Moreover, with jackknifing, both methods’ detection accuracy increases at high SNRs, where the original detection accuracy is higher than 0.5. On the other hand, jackknifing’s performance degrades at low SNR, where the accuracy is lower than half. This is in accordance with our previous analysis. Additionally, SORTE slightly outperforms VTRS.

V. Conclusion

In this paper, we proposed a novel strategy for source detection by applying jackknifing. This strategy helps the existing detection methods achieve great improvements by making full use of the limited available data. Numerical examples demonstrate the effectiveness of our strategy. For future work, we will investigate the performance effect of the percentage we choose when doing jackknifing. We will also consider applying this strategy to a co-prime array.

References