Aperture Varying Autoregressive Covariance Modeling for 2D Oversampled Receive Arrays

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***Abstract*—Recently it has been proposed that two-dimensional (2D) oversampled received arrays could be used to provide signal-to-external noise ratio (SENR) gains for over-the-horizon radar applications which are strongly externally noise limited. These array configurations can be used to exploit superdirective adaptive beamforming techniques. A key element of the superdirective adaptive beamforming process is the estimation of the array spatial noise covariance matrix. In this paper we propose a parametric covariance modeling technique called aperture varying autoregressive (AVAR) covariance modeling that captures the 2D spatial correlation structure of high-frequency (HF) background noise sampled by an oversampled 2D receive array. The use of this covariance modeling technique can significantly reduce the computational requirements for the inversion of large spatial covariance matrices. Additional gains are achieved via reduced sample support requirements for an N-element 2D receive array.**

In this paper we introduce 2D aperture varying autoregressive models AVAR(m,l) that are spatially non-stationary generalizations of traditional autoregressive AR(m) or AR (m,l) techniques. While traditional AR techniques model covariance structure as toeplitz or toeplitz-block-toeplitz, these new AVAR models enforce a banded or doubly banded inverse covariance structure which is more general. The introduced AVAR methods are closely coupled to the oversampled array architecture which in the presence of nearly homogeneous external noise exhibits spatial correlation most strongly amongst closely spaced elements. Therefore the use of these AVAR methods effectively restricts the adaptive beamforming to gains achievable through superdirective beamforming.

***Index Terms*—Over–the–horizon radar, autoregressive modeling, 2D phased array, superdirectivity, oversampled array, HF.**

I. Introduction

The problem of efficient covariance matrix estimation for adaptive beamforming problems is an ever present issue in adaptive beamforming methods. The continual advancement of sensor system architectures combined with a steady decrease in component costs allow for the possibility of receiver-per-element array designs to be utilized on the scale of 1000+ elements in a single array. Recently a 2D spatially oversampled receive array architecture [5] was proposed for a problem related to over-the-horizon radar (OTHR) detection in externally noise limited scenarios. Given the large number (300-500) of receive array elements utilized by current generation OTHR systems [6], 1000+ element systems are certainly within the realm of possibility for 2D OTHR receive arrays. In this paper we propose a parametric modeling technique for the estimation of the receive array spatial covariance matrix when the external noise covariance is derived from a nearly homogeneous background noise field as described in [5]. The use of the proposed modeling technique method allows for both reduced inverse covariance matrix computation requirements and reduced training sample support requirements.

Superdirective beamforming [7] is not a new concept. A new understanding of high frequency (HF) nighttime (3-10MHz) background noise characteristics however has resulted in a proposed 2D spatially oversampled receive array architecture that yields the potential for increased signal-to-external noise ratio (SENR) gains without the need to construct extremely large ($>$3km) 1D uniform linear array apertures. The signal processing requirement to achieve this gain is the use of adaptive beamforming. Analysis has shown that the optimal adaptive beamforming solutions are superdirective; beamforming gain is achieved by closely spacing array elements and forming beams that are more narrow in physical beam space than traditional beams. In practice the actual noise environment is both temporally non-stationary and non-homogeneous, thus the use of adaptive beamforming is critical to achieving the maximum attainable SENR gain.

In [5] several HF background noise distributions are proposed that allow one to model the potential performance of arrays with different geometries. One can show by inspection that in the case of a uniformly oversampled 2D array both the array noise covariance matrix and its inverse exhibit a banded structure, assuming the enumeration of the array elements is performed in an orderly column-wise or row-wise format. This covariance structure suggests that parametric modeling techniques may be applicable for the task of covariance matrix estimation. In this paper we explore the potential of autoregressive methods to improve the covariance matrix estimation process. In section II we introduce the autoregressive covariance model. In section III we quantify the SENR modeling loss incurred under specific array geometries and a prior noise models.

II. AVAR Modeling of 2D Oversampled HF Background Noise

The aperture varying autoregressive (AVAR) modeling methodology for covariance modeling of HF background noise sampled by an oversampled 2D receive array is based on the recently introduced 2D space-time TVAR(p,q) [1], [2] methods. These prior methods rely upon the fundamental

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band-matrix extension results of H. Dym and I. Gohberg [3]. In this paper we translate the fundamental methodology of these prior parametric modeling techniques to a 2D space-space aperture varying scenario that is relevant to recently proposed 2D oversampled receive arrays for HF OTHR [5].

Suppose a N element 2D planar receive array consists of M rows and L columns such that $N = M \times L$. The $ML \times ML$ variate covariance matrix may be presented as a block-Hermitian matrix

$$\mathbf{R} \equiv \{r_{jk}^{pq}\}_{j,k=1,\ldots,L}^{p,q=1,\ldots,M} \equiv \{\mathbf{R}_{jk}\}_{j,k=1,\ldots,L} \in H^{MLxML} \quad (1)$$

$$\mathbf{R}_{jk} \equiv \{r_{jk}^{pq}\}^{p,q=1,\ldots,M} \in C^{MxM}, \quad \mathbf{R}_{jk} = \mathbf{R}_{kj}^{H} \quad (2)$$

where C and H are the classes of complex and complex Hermitian matrices. The indices p and q refer to the array element column numbers while indices j and k refer to the array element row numbers. For a given ML-variate positive definite Hermitian-block covariance matrix $\mathbf{R}$, the 2D AVAR covariance matrix approximation of order m and l, AVAR(m,l), is the positive definite matrix $\mathbf{R}^{(m,l)}$

$$\mathbf{R}^{(m,l)} \equiv \{(r^{(m,l)})_{jk}^{pq}\}_{j,k=1,\ldots,L}^{p,q=1,\ldots,M} \quad (3)$$

$$\left[\mathbf{R}^{(m,l)}\right]^{-1} \equiv \{(f^{(m,l)})_{jk}^{pq}\}_{j,k=1,\ldots,L}^{p,q=1,\ldots,M} \quad (4)$$

that satisfies

$$\begin{aligned}(r^{(m,l)})_{jk}^{pq} &= r_{jk}^{pq} \ \ for \ |j-k| \leq l \bigcap |p-q| \leq m \equiv BB \\ (f^{(m,l)})_{jk}^{pq} &= 0 \quad for \ |j-k| > l \bigcup |p-q| > m \equiv \widetilde{BB}. \end{aligned} \quad (5)$$

This construction means that the approximation matrix $\mathbf{R}^{(m,l)}$ consists of a block band region *BB* whose element are the same as those in the specified matrix $\mathbf{R}$, while outside this region the elements are completed in a manner that ensures the inverse approximation matrix $\left[\mathbf{R}^{(m,l)}\right]^{-1}$ is strictly block banded with zeros in the region $\widetilde{BB}$. In the limiting case that m=M-1, this 2D approximation reduces to a 1D AVAR approximation AVAR$_M(l)$.

In the 1D case it can be shown that this matrix model has a unique analytic solution given by the Dym and Gohberg (DG) factorization [3], denoted by DG$^{(l)}$ $[\mathbf{R}]$ [1]. In the 2D case, existence of a unique solution can be proven, however no analytic solution exists [2]. Two methods for solving the 2D problem are introduced in [2]; in this paper we use the Alternating Dym-Gohberg algorithm, ADG. The ADG algorithm solves for the maximum entropy (ME) 2D completion through a series of alternating 1D DG factorizations.

The ML variate array covariance matrix $\mathbf{R}$ has been introduced as a Hermitian-block matrix, an $L \times L$ block matrix with $M \times M$ blocks. It is equally valid to re-order the elements so that the matrix is a $M \times M$ block matrix with $L \times L$ blocks. With the original enumeration denoted as $\mathbf{R} \equiv \mathbf{R}_{ML}$, these two specific enumerations are related as

$$\mathbf{R}_{LM} = \mathbf{J}\mathbf{R}_{ML}\mathbf{J}^{T} \quad (6)$$

where $\mathbf{J}$ is the $ML \times ML$ unitary permutation matrix with appropriate ordering of the rows of the identity matrix $\mathbf{I}_{ML}$.

ADG Algorithm [2]: Given $\mathbf{R} > 0$, compute

$$\mathbf{A}_{(m,l)}^{(1)} = DG^{(l)}\left[\mathbf{R}\right]. \quad (7)$$

Then iterate over k as

$$\mathbf{A}_{(m,l)}^{(k+1)} = DG^{(l)}\left[\mathbf{J}^{T}\left\{DG^{(m)}\left[\mathbf{J}\mathbf{A}_{(m,l)}^{(k)}\mathbf{J}^{T}\right]\right\}\mathbf{J}\right]. \quad (8)$$

Two characteristics of the 2D AVAR technique are worth emphasizing. First, significantly fewer computations are required to produce the array spatial covariance. This occurs as a result of the DG factorization which operates on small size principle minor matrices of the much larger $N \times N$ covariance $\mathbf{R}$. Secondly, also as a result of the DG factorization reduced sample support is required for covariance matrix estimation. Similar to that which is demonstrated in [2], for an $ML \times ML$ variate covariance matrix, the AVAR(m,l) minimum sample support requirement is reduced to

$$\tau_{min}^{AVAR(m,l)} = (m+1)(l+1) \quad (9)$$

in comparison to the $\tau_{min}^{SMI} = ML$ samples required for the generic sample covariance matrix estimate.

III. AVAR Superdirective SENR Modeling Loss

In this section we explore the parametric modeling loss due to the use of the non-optimal adaptive beamforming solution. For reference we repeat the form of the spatial noise model introduced in [5] as well as some of the important assumed array properties. In this paper we consider uniform rectangular arrays (URA's) composed of short non-resonant monopoles over a perfect earth. This results in an element pattern defined as

$$b_H(\theta,\phi) = 1/4(2\pi h/\lambda)^4 \sin^2(\theta). \quad (10)$$

We also consider a family of external noise distributions, homogeneous in azimuth and tapered in elevation:

$$f^{2n}(\theta,\phi) = ((1+2n)/\pi)\cos^{2n}(\theta), \ n = 0,1,2,\ldots \quad (11)$$

The external noise array spatial covariance matrix can then be specified as

$$\begin{aligned}\mathbf{R}_{ext} = \int_0^{\pi}\int_0^{2\pi} & \mathbf{s}(\theta,\phi)\mathbf{s}^{H}(\theta,\phi)\cdot \\ & b_H(\theta,\phi) f^{2n}(\theta,\phi)\sin(\theta)d\theta d\phi,\end{aligned} \quad (12)$$

where $\mathbf{s}(\theta,\phi)$ are the array steering vectors. An accurate analytic expression for (12) can be found in [5]. The complete array spatial covariance $\mathbf{R}$ is defined as

$$\mathbf{R} = \mathbf{R}_{ext} + \sigma_{int}^2\mathbf{I}. \quad (13)$$

This model formulation approximates the array internal noise covariance as white noise with power σ_{int}^2. The per element external-to-internal noise ratio (EINR) is thus defined as $\mathbf{R}_{11}/\sigma_{int}^2$.

Given a true array covariance $\mathbf{R}$ and steering vector $\mathbf{s} \equiv \mathbf{s}(\theta_0,\phi_0)$ the clairvoyant optimal beamforming is given as the well known solution

$$\mathbf{w}_{opt} = \mathbf{R}^{-1}\mathbf{s}. \quad (14)$$

The optimum array gain q_{opt} with respect to the conventional white noise processing is

$$q_{opt} = \frac{\mathbf{s}^H \mathbf{R}^{-1} \mathbf{s} \mathbf{s}^H \mathbf{R} \mathbf{s}}{|\mathbf{s}^H \mathbf{s}|^2}. \tag{15}$$

Let the AVAR(m,l) adaptive filter be defined as

$$\mathbf{w}_{(m,l)} = [\mathbf{R}^{(m,l)}]^{-1} \mathbf{s} \tag{16}$$

where $\mathbf{R}^{(m,l)}$ is the AVAR(m,l) covariance model derived from the covariance matrix $\mathbf{R}$. The SENR AVAR(m,l) model gain loss factor relative to $\mathbf{w}_{opt}$ can thus be expressed as

$$\eta = \frac{\mathbf{s}^H \mathbf{R}^{-1} \mathbf{s} \mathbf{w}_{(m,l)}^H \mathbf{R} \mathbf{w}_{(m,l)}}{|\mathbf{w}_{(m,l)}^H \mathbf{s}|^2}. \tag{17}$$

We will now explore the model mismatch loss encountered for a specific array and noise scenario. Consider a scenario with an an N =270-element rectangular array consisting of $M = 30$ columns and $L = 9$ rows. The array elements are positioned on a rectangular grid with spacing of 12.5m. The array is operated at 6MHz and thus twice oversampled. The external noise is modeled according to (11) with parameter n =1 and EINR=30dB. Figures 1 and 2 illustrate the structure of the inverse covariance for both the true covariance and an AVAR(5,5) approximation. The AVAR model does not completely capture the inverse covariance matrix structure at the large lag values, however the low lag values are qualitatively well approximated. In terms of the covariance matrix eigenspectra, figure 3 demonstrates that the AVAR approximation is quite close to the true behavior. Finally we can compare the actual performance through the realized array beampatterns, SENR gain, and the SENR loss factor. Figure 4 shows the conventional white noise beamformer beampattern. The SENR for this beamformer is 19.931dB, or 7.322dB of loss relative to the optimal solution shown in figure 5 which has an SENR of 27.253dB. The AVAR(5,5) beampattern is shown in figure 6 which has a SENR of 26.7552dB, or 0.4978dB of loss relative to the optimal solution. It is clear that the AVAR technique can efficiently model the true array covariance matrix and produce beamformer outputs with low loss.

The AVAR model mismatch loss has been demonstrated for a single choice of model parameters. It is instructive to explore how the mismatch loss behaves as the model order is varied for a fixed look direction, array configuration, and noise field. Figures 7 and 8 show the variation in the model mismatch for the same scenario described previously for two specified beam directions, (270^o,90^0) and (180^o,90^0) respectively. An important characteristic to note is the relative invariance of the mismatch loss to the model order parameter describing the band-limited approximation in the dimension orthogonal to the specified look direction. This behavior is consistent with the properties of superdirective beamforming which are sensitive to array depth for endfire geometries. In a broader context, the demonstrated mismatch loss behavior indicates that for wide azimuthal sectors of regard, the selection of nearly equal

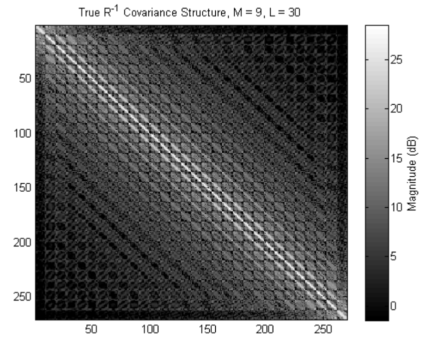

Fig. 1. Structure of true noise inverse covariance matrix for URA(9,30) and noise model n =1.

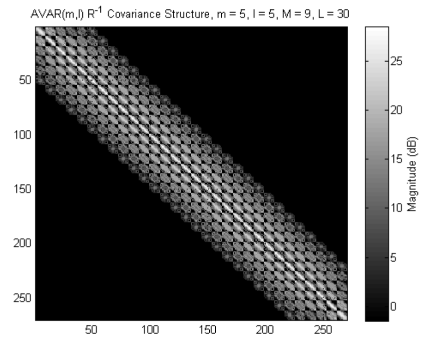

Fig. 2. Structure of AVAR(5,5) noise inverse covariance matrix for URA(9,30) and noise model n =1.

AVAR model order parameters will yield the lowest mismatch on average.

In practical applications of the AVAR(m, l) model, its parameters should be selected based on operational requirements, admissible total SENR losses, that apart from mismatch losses analyzed above, include stochastic losses associated with a limited training support, antenna calibration losses, etc. With regard to stochastic losses, in the related space-time 2D TVAR study [2], it was shown that incorrect model order selection can be compensated by lower stochastic losses in some scenarios. With regards to this study, we showed that 1) it is possible to utilize low-order models and realize low model mismatch loss and 2) joint parameter selection (m, l) should be guided by the azimuthal surveillance area of regard.

IV. Conclusion

In this paper, we conducted a theoretical analysis of a parametric covariance modeling technique called 2D AVAR in planar uniform rectangular antenna arrays (URAs), considered for receive antenna applications in OTH sky-wave radars. This study was motivated by the demonstrated theoretical SENR gain potential of such arrays, but with the recognition that efficient adaptive beamforming methods are required to realize such gains in practice due to computational and training sample support limitations. The current study has been limited to exploring the covariance model mismatch loss, however

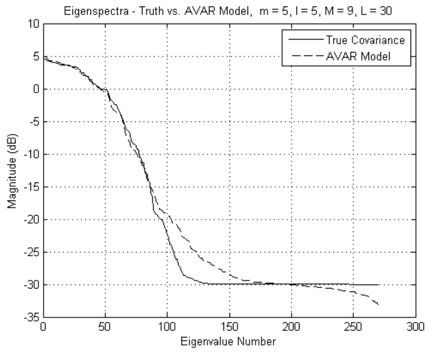

Fig. 3. Eigenspectra for true and AVAR(5,5) noise inverse covariance matrix of URA(9,30) and noise model n =1.

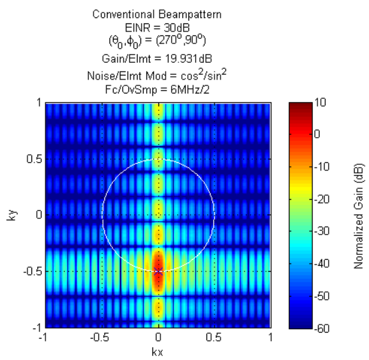

Fig. 4. Conventional white noise 2D beampattern of URA(9,30)

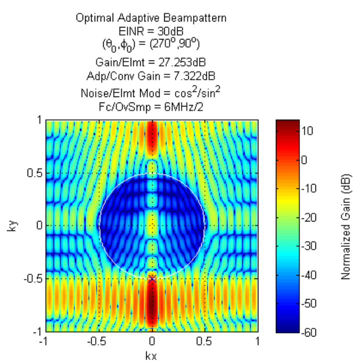

Fig. 5. Optimal 2D beampattern of URA(9,30) and noise model n =1.

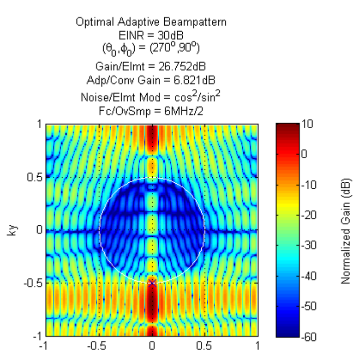

Fig. 6. AVAR(5,5) 2D beampattern of URA(9,30) and noise model n =1.

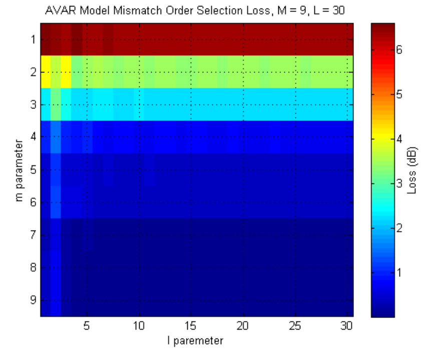

Fig. 7. AVAR Mismatch of URA(9,30) and noise model n =1 for beam steered to $(270^o,90^0)$

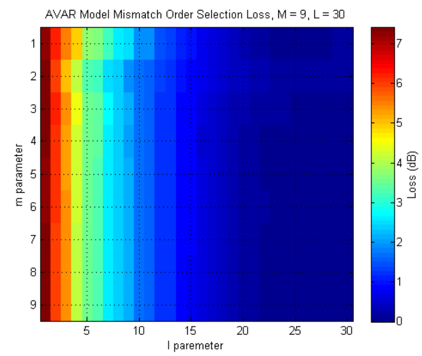

Fig. 8. AVAR Mismatch of URA(9,30) and noise model n =1 for beam steered to $(180^o,90^0)$

future efforts will demonstrate the stochastic loss aspect of the problem.

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