

# Tracking Position and Orientation of a Mobile Rigid Body

Sundeep Prabhakar Chepuri, Andrea Simonetto, Geert Leus, and Alle-Jan van der Veen  
Delft University of Technology (TU Delft), The Netherlands

**Abstract**—A framework to track the position and orientation of a moving rigid body is proposed. We consider a setup in which a few wireless sensors are mounted on a rigid body. The topology of how these sensors are mounted on the rigid body is known; however, the absolute position of the sensors or the rigid body itself is unknown. Using range-only measurements between the sensors and a few anchors (nodes with known absolute positions), and a simple kinematic model, we propose an unconstrained, unitarily-constrained, and event-triggered estimator based on Kalman filtering techniques.

## I. INTRODUCTION

Localization is one of the important tasks in wireless sensor networks (WSNs). Localization is typically based on range measurements between the sensors and a few anchors (nodes with known absolute positions). A plethora of algorithms on localization exists, an overview can be found in [1]. For non-static nodes, a variety of state-estimation algorithms to track the position of the sensors based on Kalman filters (KFs), including the extended and unscented versions of the KF have been proposed [2].

Previously, we proposed a framework for *rigid body localization* [3], [4], in which we jointly estimate the position and orientation of a *static rigid body*, without using any inertial measurements. Instead, we use only range measurements. This is useful for monitoring and maneuvering orbiting satellites, unmanned aircrafts, underwater vehicles, and robots.

Tracking the position and orientation of mobile rigid bodies is also a well-studied topic, however, they are generally treated separately [5]. While most of the existing orientation estimation methods make use of inertial measurement units (IMUs), which include sensors like accelerometers in combination with gyroscopes, positioning is typically achieved using GPS. This paper is a sequel to the rigid body localization in [3]. The main contribution of this paper is the proposed framework to track a *mobile rigid body*. More specifically, we jointly track its position and orientation. Using range measurements, and a simple kinematic model, we propose an unconstrained Kalman filter (KF), a unitarily-constrained KF (UC-KF), and an event-triggered KF (ET-KF). The proposed framework can also be used as an add-on to correct the drift errors associated with IMUs, or in environments where inertial measurements and/or positioning via GPS is not possible.

## II. PROBLEM FORMULATION

### A. System model

Consider a network with  $M$  anchors (nodes with known absolute locations) and  $N$  sensors. The sensors are mounted on the rigid body (e.g., at the factory), and the topology of

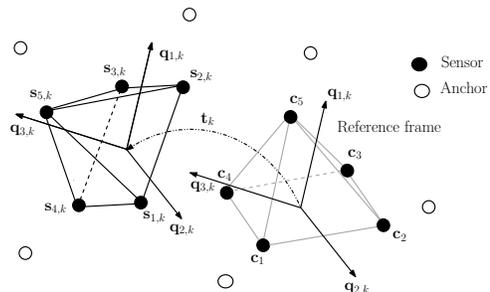


Fig. 1: A rigid body undergoing a rotation and translation.

how these sensors are mounted is known. In other words, we connect a so-called *reference frame* to the rigid body, as illustrated in Fig. 1, and in that reference frame, the coordinates of the  $n$ th sensor are given by the known  $3 \times 1$  vector  $\mathbf{c}_n = [c_{n,1}, c_{n,2}, c_{n,3}]^T$ . The sensor topology is determined by the matrix  $\mathbf{C} = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_N] \in \mathbb{R}^{3 \times N}$ . Let the absolute coordinates of the  $m$ th anchor and the  $n$ th sensor at time  $k$  be denoted by a  $3 \times 1$  vector  $\mathbf{a}_m$  and  $\mathbf{s}_{n,k}$ , respectively. These absolute positions of the anchors and the sensors (at time  $k$ ) are collected in the matrices  $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_M] \in \mathbb{R}^{3 \times M}$  and  $\mathbf{S}_k = [\mathbf{s}_{1,k}, \mathbf{s}_{2,k}, \dots, \mathbf{s}_{N,k}] \in \mathbb{R}^{3 \times N}$ , respectively. The absolute position of the sensors or the rigid body itself is not known.

The pairwise distance between the  $m$ th anchor and the  $n$ th sensor is typically obtained by ranging. The squared-range measurements between the  $m$ th anchor and the  $n$ th sensor at time  $k$  can be expressed as

$$\begin{aligned} d_{mn,k} &= \|\mathbf{a}_m - \mathbf{s}_{n,k}\|_2^2 + n_{mn,k} \\ &= \|\mathbf{a}_m\|_2^2 - 2\mathbf{a}_m^T \mathbf{s}_{n,k} + \|\mathbf{s}_{n,k}\|_2^2 + n_{mn,k}, \end{aligned} \quad (1)$$

where  $n_{mn,k}$  is the observation noise that takes into account the ranging error and the effect of squaring the range measurements. We assume  $n_{mn,k}$  as a *zero mean* random process having a variance  $\sigma_m^2 = \sigma^2 \|\mathbf{a}_m - \mathbf{s}_{1,k}\|_2^2$  [4].

### B. Known sensor geometry

A Stiefel manifold in three dimensions, commonly denoted by  $\mathcal{V}_{3,3} = \{\mathbf{Q} \in \mathbb{R}^{3 \times 3} : \mathbf{Q}^T \mathbf{Q} = \mathbf{Q} \mathbf{Q}^T = \mathbf{I}_3\}$ , is the set of all  $3 \times 3$  unitary matrices  $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \in \mathbb{R}^{3 \times 3}$ .

The absolute position of the  $n$ th sensor at time  $k$  can be written as

$$\begin{aligned} \mathbf{s}_{n,k} &= c_{n,1} \mathbf{q}_{1,k} + c_{n,2} \mathbf{q}_{2,k} + c_{n,3} \mathbf{q}_{3,k} + \mathbf{t}_k \\ &= \mathbf{Q}_k \mathbf{c}_n + \mathbf{t}_k, \end{aligned} \quad (2)$$

where  $\mathbf{t}_k \in \mathbb{R}^{3 \times 1}$  denotes the translation at time  $k$ . The combining weights  $\mathbf{c}_n$  are the *known* coordinates of the  $n$ th sensor in the reference frame, as introduced in Section II-A. The unknown unitary (rotation) matrix  $\mathbf{Q}_k \in \mathcal{V}_{3,3}$  tells us how the rigid body has rotated in the reference frame at time  $k$ .

This work was supported in part by STW under the FASTCOM project (10551) and in part by STW under the D2S2 project (10561).

As in (2), the absolute position of all the sensors at time  $k$  can be written as an affine function of the Stiefel manifold

$$\mathbf{S}_k = \mathbf{Q}_k \mathbf{C} + \mathbf{t}_k \mathbf{1}_N^T = \left[ \overbrace{\mathbf{Q}_k}^{\Theta_k} \mid \overbrace{\mathbf{t}_k}^{\mathbf{C}_e} \right] \begin{bmatrix} \mathbf{C} \\ \mathbf{1}_N^T \end{bmatrix}, \quad (3)$$

where  $\Theta_k \in \mathbb{R}^{3 \times 4}$  is the *unknown* transformation matrix. Tracking the transformation matrix is equivalent to tracking the position and orientation of the moving rigid body.

### III. STATE-SPACE MODEL

#### A. Measurement model

Collecting the squared-range measurements between the  $n$ th sensor and all the anchors, we can write (1) in a vector form as

$$\mathbf{d}_{n,k} = \mathbf{a} - 2\mathbf{A}^T \mathbf{s}_{n,k} + \|\mathbf{s}_{n,k}\|^2 \mathbf{1}_M + \mathbf{n}_{n,k}, \quad (4)$$

where  $\mathbf{d}_{n,k} = [d_{1n,k}, d_{2n,k}, \dots, d_{Mn,k}]^T \in \mathbb{R}^{M \times 1}$ ,  $\mathbf{a} = [\|\mathbf{a}_1\|^2, \|\mathbf{a}_2\|^2, \dots, \|\mathbf{a}_M\|^2]^T \in \mathbb{R}^{M \times 1}$ , and  $\mathbf{n}_{n,k} = [n_{1n,k}, \dots, n_{Mn,k}]^T \in \mathbb{R}^{M \times 1}$ . The noise covariance matrix will be  $\Sigma = \mathbb{E}\{\mathbf{n}_{n,k} \mathbf{n}_{n,k}^T\} = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_M^2) \in \mathbb{R}^{M \times M}$ .

We eliminate the vector  $\|\mathbf{s}_{n,k}\|^2 \mathbf{1}_M$  in (4) using an  $M \times (M-1)$  isometry decomposition of  $\mathbf{P}_M = \mathbf{I}_M - \mathbf{1}_M \mathbf{1}_M^T / M = \mathbf{U}_M \mathbf{U}_M^T$ , such that  $\mathbf{U}_M^T \mathbf{1}_M = \mathbf{0}_{M-1}$ . Pre-multiplying both sides of (4), we arrive at

$$\mathbf{U}_M^T (\mathbf{d}_{n,k} - \mathbf{a}) = -2\mathbf{U}_M^T \mathbf{A}^T \mathbf{s}_{n,k} + \mathbf{U}_M^T \mathbf{n}_{n,k}. \quad (5)$$

We next whiten (5) by multiplying both sides of (5) with a pre-whitening matrix  $\mathbf{W} \in \mathbb{R}^{(M-1) \times (M-1)}$ , which leads to

$$\mathbf{W} \mathbf{U}_M^T (\mathbf{d}_{n,k} - \mathbf{a}) = -2\mathbf{W} \mathbf{U}_M^T \mathbf{A}^T \mathbf{s}_{n,k} + \mathbf{W} \mathbf{U}_M^T \mathbf{n}_{n,k}, \quad (6)$$

where we define  $\mathbf{W}$  such that  $\mathbf{W} (\mathbf{U}_M^T \Sigma \mathbf{U}_M) \mathbf{W} = \mathbf{I}_{M-1}$ . Stacking (6) for all the  $N$  sensors, we obtain

$$\mathbf{W} \mathbf{U}_M^T \mathbf{D}_k = -2\mathbf{W} \mathbf{U}_M^T \mathbf{A}^T \mathbf{S}_k + \mathbf{W} \mathbf{U}_M^T \mathbf{N}_k, \quad (7)$$

where  $\mathbf{D}_k = [\mathbf{d}_{1,k}, \mathbf{d}_{2,k}, \dots, \mathbf{d}_{N,k}] - \mathbf{a} \mathbf{1}_N^T \in \mathbb{R}^{M \times N}$ , and  $\mathbf{N}_k = [\mathbf{n}_{1,k}, \mathbf{n}_{2,k}, \dots, \mathbf{n}_{N,k}] \in \mathbb{R}^{M \times N}$ . Substituting (3) in (7) we arrive at the following linear observation model

$$\mathbf{W} \mathbf{U}_M^T \mathbf{D}_k = -2\mathbf{W} \mathbf{U}_M^T \mathbf{A}^T \Theta_k \mathbf{C}_e + \mathbf{W} \mathbf{U}_M^T \mathbf{N}_k$$

which can be written as

$$\bar{\mathbf{D}}_k = \bar{\mathbf{A}} \Theta_k \mathbf{C}_e + \bar{\mathbf{N}}_k, \quad (8)$$

where  $\bar{\mathbf{D}}_k = \mathbf{W} \mathbf{U}_M^T \mathbf{D}_k \in \mathbb{R}^{(M-1) \times N}$ ,  $\bar{\mathbf{A}} = -2\mathbf{W} \mathbf{U}_M^T \mathbf{A}^T \in \mathbb{R}^{(M-1) \times 3}$ , and  $\bar{\mathbf{N}}_k = \mathbf{W} \mathbf{U}_M^T \mathbf{N}_k \in \mathbb{R}^{(M-1) \times N}$ . Vectorizing (8) leads to the measurement model

$$\bar{\mathbf{d}}_k = \mathbf{H} \theta_k + \bar{\mathbf{n}}_k, \quad (9)$$

where  $\mathbf{H} = (\mathbf{C}_e^T \otimes \bar{\mathbf{A}}) \in \mathbb{R}^{N(M-1) \times 12}$  ( $\otimes$  is the Kronecker product),  $\bar{\mathbf{d}}_k = \text{vec}(\bar{\mathbf{D}}_k) \in \mathbb{R}^{(M-1)N \times 1}$ ,  $\bar{\mathbf{n}}_k = \text{vec}(\bar{\mathbf{N}}_k) \in \mathbb{R}^{(M-1)N \times 1}$ , and the unknown  $12 \times 1$  vector  $\theta_k$  is given by

$$\theta_k = \text{vec}(\Theta_k) = [\mathbf{q}_k^T, \mathbf{t}_k^T]^T \quad \text{with} \quad \mathbf{q}_k = \text{vec}(\mathbf{Q}_k).$$

The covariance matrix of  $\bar{\mathbf{n}}_k$  will be  $\mathbb{E}\{\bar{\mathbf{n}}_k \bar{\mathbf{n}}_k^T\} = \mathbf{I}_{(M-1)N}$  [3].

#### B. Kinematic model

We consider a simple kinematic model for the motion of rigid bodies, in which the motion of the body is defined by the following two parameters: (a) the rate of change of rotation which depends on the *angular velocity*, and (b) the rate of change of translation which depends on the *linear velocity*.

The kinematic rotation update equation is given by

$$\mathbf{Q}_k = \mathbf{F}_{Q,k} \mathbf{Q}_{k-1} \quad (10)$$

where the matrix  $\mathbf{F}_{Q,k} \in \mathcal{V}_{3,3}$  has a closed-form solution following from the *Rodrigues formula* [6, pg. 370]:

$$\begin{aligned} \mathbf{F}_{Q,k} &= \exp(\Omega_k) \\ &= \mathbf{I}_3 + \frac{\Omega_k}{\|\tau_s \boldsymbol{\omega}_k\|_2} \sin(\|\tau_s \boldsymbol{\omega}_k\|_2) + \frac{\Omega_k^2}{\|\tau_s \boldsymbol{\omega}_k\|_2^2} (1 - \cos(\|\tau_s \boldsymbol{\omega}_k\|_2)), \end{aligned}$$

where  $\tau_s$  is the sampling time, and the skew-symmetric matrix

$$\Omega_k = \begin{bmatrix} 0 & -\omega_{3,k} & \omega_{2,k} \\ \omega_{3,k} & 0 & -\omega_{1,k} \\ -\omega_{2,k} & \omega_{1,k} & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3},$$

is the angular velocity matrix constructed from the angular velocity vector  $\boldsymbol{\omega}_k = [\omega_{1,k}, \omega_{2,k}, \omega_{3,k}]^T \in \mathbb{R}^{3 \times 1}$ . Since the matrix  $\mathbf{F}_{Q,k} \in \mathcal{V}_{3,3}$ , the following is always true:  $\mathbf{Q}_k \in \mathcal{V}_{3,3}$ , if and only if,  $\mathbf{Q}_{k-1} \in \mathcal{V}_{3,3}$ . The angular velocity dynamics in (10) is usually perturbed (due to slight speed corrections, for instance). Taking these perturbations into account the rotation update equation in (10) can be written as

$$\mathbf{Q}_k = \mathbf{F}'_{Q,k} \mathbf{Q}_{k-1} \quad (11)$$

where  $\mathbf{F}'_{Q,k} = \exp(\Omega_k + \mathbf{E}_k)$  with a skew-symmetric matrix  $\mathbf{E}_k$  constructed using  $\mathbf{e}_k$  that correspond to the perturbations on  $\boldsymbol{\omega}_k$ . Here, we assume that  $\mathbb{E}\{\mathbf{e}_k \mathbf{e}_k^T\} = \sigma_\omega^2 \mathbf{I}_3$ . The process noise depends on the state variable, hence, we use an approximate model

$$\mathbf{Q}_k = \mathbf{F}_{Q,k} \mathbf{Q}_{k-1} + \mathbf{Z}_{Q,k}, \quad (12)$$

which when vectorized leads to

$$\mathbf{q}_k = (\mathbf{I}_3 \otimes \mathbf{F}_{Q,k}) \mathbf{q}_{k-1} + \mathbf{z}_{q,k}, \quad (13)$$

where  $\mathbf{z}_{q,k} = \text{vec}(\mathbf{Z}_{Q,k})$  is the process noise with  $\mathbb{E}\{\mathbf{z}_{q,k} \mathbf{z}_{q,k}^T\} = \sigma_q^2 \mathbf{I}_9$ .

The update equation for the translational displacement is given by

$$\mathbf{t}_k = \mathbf{t}_{k-1} + \tau_s \dot{\mathbf{t}}_k + \mathbf{z}_{t,k} \quad (14)$$

where  $\dot{\mathbf{t}}_k$  is the linear velocity which is subject to some random noise denoted by  $\mathbf{z}_{t,k}$  with  $\mathbb{E}\{\mathbf{z}_{t,k} \mathbf{z}_{t,k}^T\} = \sigma_t^2 \mathbf{I}_3$ . Stacking (13) and (14), we arrive at the kinematic model

$$\theta_k = \mathbf{F}_k \theta_{k-1} + \mathbf{u}_k + \mathbf{z}_k, \quad (15)$$

where  $\mathbf{F}_k = \text{diag}(\mathbf{I}_3 \otimes \mathbf{F}_{Q,k}, \mathbf{I}_3) \in \mathbb{R}^{12 \times 12}$  is the block diagonal state-transition matrix,  $\mathbf{u}_k = \tau_s [\mathbf{0}_9^T, \dot{\mathbf{t}}_k^T]^T \in \mathbb{R}^{12 \times 1}$  is the known control vector, and  $\mathbf{z}_k = [\mathbf{z}_{q,k}^T, \mathbf{z}_{t,k}^T]^T \in \mathbb{R}^{12 \times 1}$  is the process noise with a block diagonal covariance matrix  $\mathbf{M} = \mathbb{E}\{\mathbf{z}_k \mathbf{z}_k^T\} = \text{diag}(\sigma_q^2 \mathbf{I}_9, \sigma_t^2 \mathbf{I}_3)$ .

Recalling (15) and (9), we have the *state-space model*

$$\theta_k = \mathbf{F}_k \theta_{k-1} + \mathbf{u}_k + \mathbf{z}_k, \quad (16a)$$

$$\bar{\mathbf{d}}_k = \mathbf{H} \theta_k + \bar{\mathbf{n}}_k, \quad (16b)$$

where we assume that the dynamic model is known.

## IV. PROPOSED ESTIMATORS

## A. Unconstrained Kalman filter (KF)

For the state-space model in (16), the derivation of an (unconstrained) KF is well-known [2]. Assuming that the estimate  $\hat{\boldsymbol{\theta}}_{k-1}$  and the error covariance matrix  $\mathbb{E}\{(\hat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}_{k-1})(\hat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}_{k-1})^T\} = \mathbf{P}_{k-1}$  are available from the previous time step  $k-1$ , then at time  $k$ , the KF state predictor and its error covariance are obtained as

$$\hat{\boldsymbol{\theta}}_{k|k-1} = \mathbf{F}_k \hat{\boldsymbol{\theta}}_{k-1} + \mathbf{u}_k \quad (17a)$$

$$\begin{aligned} \mathbf{P}_{k|k-1} &= \mathbb{E}\{(\hat{\boldsymbol{\theta}}_{k|k-1} - \boldsymbol{\theta}_k)(\hat{\boldsymbol{\theta}}_{k|k-1} - \boldsymbol{\theta}_k)^T\} \\ &= \mathbf{F}_k \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{M}. \end{aligned} \quad (17b)$$

We can view  $\hat{\boldsymbol{\theta}}_{k|k-1}$  in (17a) as an additional noisy measurement of  $\boldsymbol{\theta}_k$ , such that

$$\begin{bmatrix} \hat{\boldsymbol{\theta}}_{k|k-1} \\ \hat{\mathbf{q}}_{k|k-1} \\ \hat{\mathbf{t}}_{k|k-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_k \\ \mathbf{q}_k \\ \mathbf{t}_k \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{k|k-1} \\ \mathbf{e}_{q,k|k-1} \\ \mathbf{e}_{t,k|k-1} \end{bmatrix} \quad (18)$$

where

$$\mathbb{E}\{\mathbf{e}_{k|k-1} \mathbf{e}_{k|k-1}^T\} = \mathbf{P}_{k|k-1} = \text{diag}(\mathbf{P}_{q,k|k-1}, \mathbf{P}_{t,k|k-1}).$$

Here,  $\mathbf{P}_{q,k|k-1}$  and  $\mathbf{P}_{t,k|k-1}$  relate to  $\mathbf{e}_{q,k|k-1}$  and  $\mathbf{e}_{t,k|k-1}$ , respectively. Stacking (16b) and (18), we form an augmented measurement vector

$$\begin{bmatrix} \hat{\boldsymbol{\theta}}_{k|k-1} \\ \hat{\mathbf{d}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{12} \\ \mathbf{H} \end{bmatrix} \boldsymbol{\theta}_k + \begin{bmatrix} \mathbf{e}_{k|k-1} \\ \tilde{\mathbf{n}}_k \end{bmatrix} \quad (19)$$

where the augmented noise vector has a block diagonal covariance matrix  $\text{diag}(\mathbf{P}_{k|k-1}, \mathbf{I})$ . The (unconstrained) KF is obtained by solving (19) in the weighted least-squares (WLS) sense, i.e., solving the optimization problem

$$\hat{\boldsymbol{\theta}}_k = \arg \min_{\boldsymbol{\theta}_k} \|\mathbf{P}_{k|k-1}^{-1/2} (\hat{\boldsymbol{\theta}}_{k|k-1} - \boldsymbol{\theta}_k)\|_2^2 + \|\tilde{\mathbf{d}}_k - \mathbf{H}\boldsymbol{\theta}_k\|_2^2, \quad (20)$$

whose solution is commonly referred to as the KF update equation

$$\hat{\boldsymbol{\theta}}_k = \hat{\boldsymbol{\theta}}_{k|k-1} + \mathbf{K}_k (\tilde{\mathbf{d}}_k - \mathbf{H}\hat{\boldsymbol{\theta}}_{k|k-1})^{-1} \quad (21)$$

where  $\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}^T (\mathbf{H} \mathbf{P}_{k|k-1} \mathbf{H}^T + \mathbf{I})^{-1}$  is the Kalman gain, and  $\text{vec}^{-1}(\hat{\boldsymbol{\theta}}_k) = [\hat{\mathbf{Q}}_k | \hat{\mathbf{t}}_k]$ . The estimate has an error covariance

$$\begin{aligned} \mathbf{P}_k &= \mathbb{E}\{(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)^T\} = \text{diag}(\mathbf{P}_{q,k}, \mathbf{P}_{t,k}) \\ &= (\mathbf{I}_{12} - \mathbf{K}_k \mathbf{H}) \mathbf{P}_{k|k-1}, \end{aligned} \quad (22)$$

where  $\mathbf{P}_{q,k}$  and  $\mathbf{P}_{t,k}$  relate to the error covariance of  $\hat{\mathbf{q}}_k$  and  $\hat{\mathbf{t}}_k$ , respectively. However, generally the solution  $\hat{\mathbf{Q}}_k \notin \mathcal{V}_{3,3}$ . In order to ensure that the estimated  $\hat{\mathbf{Q}}_k$  is a unitary matrix, we propose two extensions.

## B. Unitarily-constrained Kalman filter (UC-KF)

The UC-KF optimization problem is obtained by adding a unitary constraint to (20) leading to the following unitarily-constrained WLS (UC-WLS) problem

$$\begin{aligned} \hat{\boldsymbol{\theta}}_k &= \arg \min_{\boldsymbol{\theta}_k} \|\mathbf{P}_{k|k-1}^{-1/2} (\hat{\boldsymbol{\theta}}_{k|k-1} - \boldsymbol{\theta}_k)\|_2^2 + \|\tilde{\mathbf{d}}_k - \mathbf{H}\boldsymbol{\theta}_k\|_2^2 \\ \text{s.t. } \mathbf{Q}_k^T \mathbf{Q}_k &= \mathbf{I}_3. \end{aligned} \quad (23)$$

In order to solve the UC-WLS problem, we decouple the rotation and translation collected in  $\boldsymbol{\theta}_k = [\mathbf{q}_k^T, \mathbf{t}_k^T]^T$  in (18) and (16b) that corresponds to the first and the second term of (23), respectively.

Let us define an  $N \times (N-1)$  isometry matrix  $\mathbf{U}_N$  obtained by collecting orthonormal basis vectors of the null-space of  $\mathbf{1}_N$  such that  $\mathbf{1}_N^T \mathbf{U}_N = \mathbf{0}_{N-1}^T$ . Recall that (16b) is a vectorized version of (8), and by right-multiplying both sides of (8) with  $\mathbf{U}_N$  we can project out the row vector  $\mathbf{1}_N^T$  in  $\mathbf{C}_e$ , and hence, decouple rotation from translation. Right-multiplying  $\mathbf{U}_N$  on both sides of (8) we get

$$\tilde{\mathbf{D}}_k = \bar{\mathbf{A}} \mathbf{Q}_k \bar{\mathbf{C}} + \tilde{\mathbf{N}}_k,$$

where  $\tilde{\mathbf{D}}_k = \bar{\mathbf{D}}_k \mathbf{U}_N \in \mathbb{R}^K$  with  $K = (M-1) \times (N-1)$ ,  $\bar{\mathbf{C}} = \mathbf{C} \mathbf{U}_N \in \mathbb{R}^{3 \times (N-1)}$ , and  $\tilde{\mathbf{N}}_k = \tilde{\mathbf{N}}_k \mathbf{U}_N \in \mathbb{R}^K$ . This can be then vectorized as

$$\tilde{\mathbf{d}}_k = \bar{\mathbf{H}} \mathbf{q}_k + \tilde{\mathbf{n}}_k, \quad (24)$$

where  $\bar{\mathbf{H}} = \bar{\mathbf{C}}^T \otimes \bar{\mathbf{A}}$ ,  $\tilde{\mathbf{d}}_k = \text{vec}(\tilde{\mathbf{D}}_k)$ , and  $\tilde{\mathbf{n}}_k = \text{vec}(\tilde{\mathbf{N}}_k)$  with  $\mathbb{E}\{\tilde{\mathbf{n}}_k \tilde{\mathbf{n}}_k^T\} = \mathbf{I}_{(M-1)(N-1)}$ . As before, we augment the first sub-row of (18) and (24) to get

$$\begin{bmatrix} \hat{\mathbf{q}}_{k|k-1} \\ \tilde{\mathbf{d}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{I}_9 \\ \bar{\mathbf{H}} \end{bmatrix} \mathbf{q}_k + \begin{bmatrix} \mathbf{e}_{q,k|k-1} \\ \tilde{\mathbf{n}}_k \end{bmatrix} \quad (25)$$

The optimization problem in (25) is solved using an iterative algorithm based on Newton's method described in [3], which solves the following optimization problem over a Stiefel manifold

$$\begin{aligned} \hat{\mathbf{q}}_{k,UC} &= \arg \min_{\mathbf{Q}_k} \|\mathbf{P}_{q,k|k-1}^{-1/2} (\hat{\mathbf{q}}_{k|k-1} - \mathbf{q}_k)\|_2^2 + \|\tilde{\mathbf{d}}_k - \bar{\mathbf{H}} \mathbf{q}_k\|_2^2 \\ \text{s.t. } \mathbf{Q}_k &\in \mathcal{V}_{3,3} \end{aligned}$$

which is equivalent to solving the problem [3]

$$\hat{\mathbf{q}}_{k,UC} = \arg \min_{\mathbf{q}_k} \|f(\mathbf{q}_k) - \mathbf{b}\|_2^2 \quad \text{s.t. } \mathbf{Q}_k \in \mathcal{V}_{3,3}, \quad (26)$$

where we use the following  $(K+9) \times 1$  vectors

$$f(\mathbf{q}_k) := \begin{bmatrix} \mathbf{P}_{q,k|k-1}^{-1/2} \\ \bar{\mathbf{H}} \end{bmatrix} \mathbf{q}_k, \quad \text{and} \quad \mathbf{b} := \begin{bmatrix} \mathbf{P}_{q,k|k-1}^{-1/2} \mathbf{q}_{k|k-1} \\ \tilde{\mathbf{d}} \end{bmatrix}.$$

Substituting the solution  $\hat{\mathbf{Q}}_{k,UC} = \text{vec}^{-1}(\hat{\mathbf{q}}_{k,UC})$  of (26) in (8), we get the residual  $\tilde{\mathbf{D}}_k - \bar{\mathbf{A}} \hat{\mathbf{Q}}_{k,UC} \bar{\mathbf{C}} = \bar{\mathbf{A}} \mathbf{t}_k \mathbf{1}_N^T + \tilde{\mathbf{N}}_k$ . Augmenting the vectorized version of this residual and the second sub-row of (18), we get

$$\begin{bmatrix} \hat{\mathbf{t}}_{k|k-1} \\ \bar{\mathbf{r}}_t \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 \\ \mathbf{1}_N \otimes \bar{\mathbf{A}} \end{bmatrix} \mathbf{t}_k + \begin{bmatrix} \mathbf{e}_{t,k|k-1} \\ \tilde{\mathbf{n}}_k \end{bmatrix}, \quad (27)$$

where  $\bar{\mathbf{r}}_t = \text{vec}(\tilde{\mathbf{D}}_k - \bar{\mathbf{A}} \hat{\mathbf{Q}}_{k,UC} \bar{\mathbf{C}})$  is the residual vector. The UC-WLS estimate of the translation is obtained by solving a standard WLS problem

$$\begin{aligned} \hat{\mathbf{t}}_{k,UC} &= \arg \min_{\mathbf{t}_k} \|\mathbf{P}_{t,k|k-1}^{-1/2} (\hat{\mathbf{t}}_{k|k-1} - \mathbf{t}_k)\|_2^2 \\ &\quad + \|\bar{\mathbf{r}}_t - (\mathbf{1}_N \otimes \bar{\mathbf{A}}) \mathbf{t}_k\|_2^2. \end{aligned}$$

Note that in the UC-KF, we still use (22) for updating the error covariance matrix and as such the covariance matrix is overestimated in case of the UC-KF.

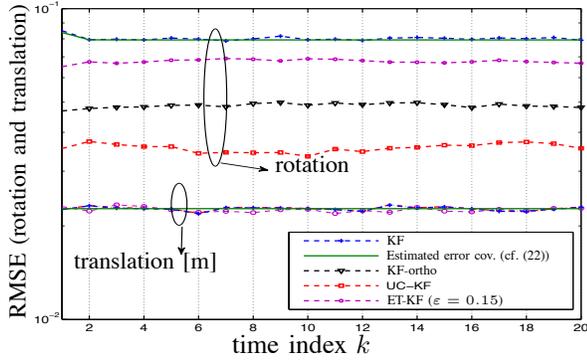


Fig. 2: RMSE of the estimated rotation and translation.

### C. Event-triggered KF-orthonormalization (ET-KF)

Even though the UC-KF is the desired estimator, it has the disadvantage of computational complexity, since it involves several Newton iterations for solving the UC-WLS at each time step.

Since one expects that, for low noise/high sampling rate settings, the unitarity conditions will hold approximately for a few sampling intervals, a more reasonable approach is to use an event-triggered methodology. Event-triggering is a growing research topic in different fields, such as control, estimation, and optimization [7]. For our estimation problem, event-triggering means that we can decide whether to orthonormalize the solution  $\hat{\mathbf{Q}}_k$  from the KF based on the predefined threshold  $\|\hat{\mathbf{Q}}_k^T \hat{\mathbf{Q}}_k - \mathbf{I}_3\|_F > \varepsilon$ .

In event-triggered KF-orthonormalization (ET-KF) at each time step, the (unconstrained) KF is first solved. Then we evaluate  $\|\hat{\mathbf{Q}}_k^T \hat{\mathbf{Q}}_k - \mathbf{I}_3\|_F$ , and if it is bigger than  $\varepsilon$ , we orthonormalize  $\hat{\mathbf{Q}}_k$  by solving the following special case of the *orthogonal Procrustes problem* [3]

$$\begin{aligned} \hat{\mathbf{Q}}_{k,ET} &= \arg \min_{\mathbf{Q}_k} \|\mathbf{Q}_k - \hat{\mathbf{Q}}_k\|_F^2 \quad \text{s.t.} \quad \mathbf{Q}_k \in \mathcal{V}_{3,3} \\ &= (\hat{\mathbf{Q}}_k \hat{\mathbf{Q}}_k^T)^{-1/2} \hat{\mathbf{Q}}_k. \end{aligned} \quad (28)$$

Again, we use (22) for updating the error covariance matrix. In this way, we can trade-off estimation accuracy and computational complexity.

## V. SIMULATIONS

We consider a square based pyramid of size  $1(l) \times 1(w) \times 1(h)$  m for the rigid body with  $N = 5$  sensors mounted on the vertices of the considered pyramid. Four anchors are deployed uniformly at random within a range of 2 km. The simulations are averaged over  $N_{\text{exp}} = 1000$  independent Monte-Carlo trials. We use the following parameters in the simulations:  $\sigma = 0.0316$  m,  $\sigma_t = 1$  m,  $\sigma_\omega = \sigma_{q_i} = 0.1$  deg/s. We assume a constant velocity dynamics with  $\mathbf{t}_k = \mathbf{1}_3$  m/s and  $\omega_k = \mathbf{1}_3$  deg/s. In each Monte-Carlo trial, the state estimates are initialized with  $\hat{\mathbf{Q}}_{-1} = \mathbf{I}_3$ ,  $\hat{\mathbf{t}}_{-1} = \mathbf{0}_3$ , and a high error covariance of  $\mathbf{P}_{-1} = 100\mathbf{I}_{12}$ . A trajectory of 20 time steps is generated during each trial according to the kinematic model. The performances of the proposed estimators are compared in terms of the root mean square error (RMSE) of  $\mathbf{t}_k$  defined as  $\text{RMSE}(\mathbf{t}_k) = \sqrt{\frac{1}{N_{\text{exp}}} \sum_{n=1}^{N_{\text{exp}}} \|\mathbf{t}_k - \hat{\mathbf{t}}_k^{(n)}\|_2^2}$ ,

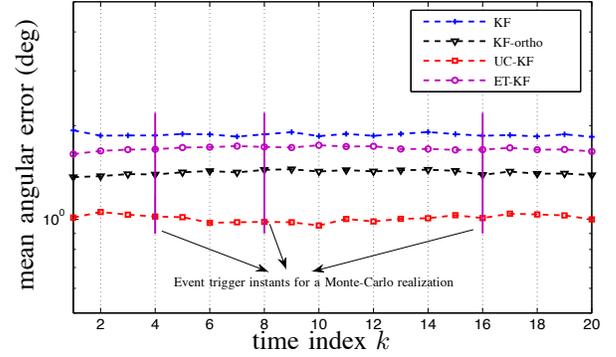


Fig. 3: Mean angular error of the orientation estimates.

where  $\hat{\mathbf{t}}_k^{(n)}$  denotes the estimate obtained during the  $n$ -th Monte-Carlo experiment. To analyze the performance of the rotation estimates, we define one more metric called the mean angular error (MAE) defined as  $\text{MAE}(\mathbf{Q}_k) = \sqrt{\frac{1}{3N_{\text{exp}}} \sum_{n=1}^{N_{\text{exp}}} \text{tr}(\arccos(\mathbf{Q}_k^T \hat{\mathbf{Q}}_{k,\text{norm}}^{(n)}))}$ , where we normalize the columns of  $\hat{\mathbf{Q}}_k^{(n)}$  as  $\hat{\mathbf{Q}}_{k,\text{norm}}^{(n)} = [\frac{\hat{\mathbf{q}}_{k,1}}{\|\hat{\mathbf{q}}_{k,1}\|_2}, \frac{\hat{\mathbf{q}}_{k,2}}{\|\hat{\mathbf{q}}_{k,2}\|_2}, \frac{\hat{\mathbf{q}}_{k,3}}{\|\hat{\mathbf{q}}_{k,3}\|_2}]$  when  $\hat{\mathbf{Q}}_k^{(n)} \notin \mathcal{V}_{3,3}$ , and  $\hat{\mathbf{Q}}_k^{(n)}$  and  $\hat{\mathbf{Q}}_{k,\text{norm}}^{(n)}$  correspond to the estimate obtained during the  $n$ th Monte-Carlo trial. The estimated RMSE of rotation and translation is computed using  $\sqrt{\frac{1}{N_{\text{exp}}} \sum_{n=1}^{N_{\text{exp}}} \text{tr}(\mathbf{P}_{q,k}^{(n)})}$  and  $\sqrt{\frac{1}{N_{\text{exp}}} \sum_{n=1}^{N_{\text{exp}}} \text{tr}(\mathbf{P}_{t,k}^{(n)})}$ , respectively, where  $\mathbf{P}_{q,k}^{(n)}$  and  $\mathbf{P}_{t,k}^{(n)}$  correspond to the error covariance during the  $n$ th Monte-Carlo trial (see solid green line in Fig. 2).

In Fig. 2, the RMSE of the rotation estimates is illustrated for the proposed estimators, along with the estimated covariance. The RMSE of the unconstrained KF follows the estimated covariance. However, the constrained estimators have a lower RMSE (see Fig. 3 for the MAE). The performance of the translation estimates for the considered scenario is similar for all methods. The RMSE of the ET-KF estimate for  $\varepsilon = 0.15$  is higher than orthonormalizing at each step (indicated as KF-ortho).

## REFERENCES

- [1] S. Gezici, Z. Tian, G.B. Giannakis, H. Kobayashi, A. F. Molisch, H. V. Poor, and Z. Sahinoglu, "Localization via ultra-wideband radios: a look at positioning aspects for future sensor networks," *IEEE Signal Process. Mag.*, vol. 22, no. 4, pp. 70–84, July 2005.
- [2] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation with applications to tracking and navigation: theory algorithms and software*, John Wiley & Sons, 2004.
- [3] S. P. Chepuri, G. Leus, and A.-J. van der Veen, "Rigid body localization using sensor networks: Position and orientation estimation," *IEEE Trans. Signal Process.* (available on arXiv), Jul. 2013.
- [4] S. P. Chepuri, G. Leus, and A.-J. van der Veen, "Position and orientation estimation of a rigid body: Rigid body localization," in *Proc. of IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, May 2013.
- [5] N. A. Chaturvedi, A. K. Sanyal, and N. H. Mc Clamroch, "Rigid-body attitude control," *IEEE Control Syst. Mag.*, vol. 31, no. 3, pp. 30–51, Jun. 2011.
- [6] S. Sastry, *Nonlinear Systems: Analysis, Stability and Control*, Interdisciplinary applied mathematics: Systems and control. Springer, 1999.
- [7] M. Lemmon, *Networked Control Systems*, vol. 405 of *Lecture Notes in Control and Information Sciences*, chapter Event-triggered Feedback in Control, Estimation, and Optimization, pp. 293 – 358, Springer, 2010.