Enhanced Capon Beamformer Using Regularized Covariance Matching

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Abstract—**The Capon method is a powerful nonparametric approach in array processing based on the sample covariance matrix. For small sample sets, however, its performance is degraded. In this paper we formulate a regularized covariance matching framework based on the nuclear norm for enhancing the Capon method. An approximate iterative solution is developed and tested using simulated data. Appropriate regularization parameter values are also inferred from the data, drawing upon the cross-validation approach. The results show significantly improved spatial spectral and signal waveform estimates.**

I. Introduction

In array processing, nonparametric approaches offer a certain advantage over high-resolution parametric approaches, such as MUSIC or ESPRIT, in that the former do not require specification of the correct number of emitting sources or the precise covariance structure of the data [1]. Instead, the nonparametric approaches, such as beamforming and the Capon method [2], provide a spatial power spectrum which contains crucial information about the number of sources, their signal powers, and directions of arrival (DOA) through distinct spectral peaks. In practice, however, the Capon method relies on a finite sample covariance matrix estimate of the array output. This results in biased power estimates [3] and errors of estimated signal waveforms [4], [5]. Small sample sets are advantageous as they require less acquisition time and less restrictive assumptions on the stationary of the signals.

In this paper we aim to enhance the Capon method for small-sample and low signal to noise (SNR) ratio scenarios by an improved covariance matrix estimator that exploits the unknown low-rank signal structure. The nuclear norm of a matrix provides a tractable envelope function of its rank, and can therefore be used to solve intractable rank minimization problems as shown in [6]. This enables the formulation of a regularized covariance matching problem which is solved approximately by an iteratively reweighted technique, cf. [7], [8], [9]. The resulting covariance matrix estimate and the Capon beamformer are evaluate by means of simulations.

Notation: The nuclear norm is given by the sum of singular values $\|\mathbf{X}\|_* = \sum_i \sigma_i$. If $\mathbf{Q}$ is positive definite then the weighted norm is $\|\mathbf{x}\|_{\mathbf{Q}} = \sqrt{\mathbf{x}^*\mathbf{Q}\mathbf{x}}$ and we denote a matrix square root by $\mathbf{Q}^{*/2}\mathbf{Q}^{1/2}$. The invertible vectorization and matrix construction mappings are denoted $\text{vec}(\cdot) : \mathbb{C}^{n\times p} \to \mathbb{C}^{np\times 1}$ and $\text{mat}_{n,p}(\cdot) : \mathbb{C}^{np\times 1} \to \mathbb{C}^{n\times p}$, respectively.

II. Problem formulation

The Capon method formulates a linear filter $\mathbf{w}(\theta)$ for an m-element array output $\mathbf{y}(t) \in \mathbb{C}^m$ such that a signal from a given direction θ passes undistorted, while attenuating signals from all other angles as much as possible [2], [1]. Let $\mathbf{a}(\theta) \in \mathbb{C}^m$ denote the array response vector, then $\mathbf{w}$ is designed to minimize the the output power $\mathrm{E}[|\mathbf{w}^*\mathbf{y}(t)|^2]$ subject to $\mathbf{w}^*\mathbf{a}(\theta) = 1$. The optimum linear filter equals $\mathbf{w}(\theta) = \mathbf{R}^{-1}\mathbf{a}(\theta)/(\mathbf{a}^*(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta))$ and the power of the filtered signal becomes

$$P(\theta) = \frac{1}{\mathbf{a}^*(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)}, \tag{1}$$

which provides a spatial power spectrum. Here $\mathbf{R} \succ \mathbf{0}$ denotes the $m\times m$ covariance matrix of $\mathbf{y}(t)$. In practice $\mathbf{R}$ is estimated using T snapshots $\{\mathbf{y}(t)\}_{t=1}^T$.

In the following we assume that $\mathbf{y}(t)$ is obtained from a uniform linear array and modeled as

$$\mathbf{y}(t) = \sum_{i=1}^{d} \mathbf{a}(\theta_i)s_i(t) + \mathbf{n}(t) \in \mathbb{C}^m, \tag{2}$$

where the number of sources $d < m$ is *unknown*. The source signals $\{s_i(t)\}_{i=1}^d$, with DOAs $\{\theta_i\}_{i=1}^d$, are uncorrelated and the zero-mean noise is spatially white $\mathrm{E}[\mathbf{n}(t)\mathbf{n}^*(t)] = \sigma^2\mathbf{I}_m$. The goal is to estimate the covariance matrix $\mathbf{R}$ for the Capon method.

III. Regularized covariance matching

Given signal model (2) we have that

$$\mathbf{R} = \mathbf{A}\mathbf{P}\mathbf{A}^* + \sigma^2\mathbf{I}_m, \tag{3}$$

where $\mathbf{P} = \text{diag}(\mathrm{E}[|s_1(t)|^2], \dots, \mathrm{E}[|s_d(t)|^2])$ and $\mathbf{A} = [\mathbf{a}(\theta_1) \ \cdots \ \mathbf{a}(\theta_d)]$ is a Vandermonde matrix. Therefore the signal component $\mathbf{X} \triangleq \mathbf{A}\mathbf{P}\mathbf{A}^*$ is a Hermitian Toeplitz matrix of unknown rank d. We propose to estimate $\mathbf{R} = \mathbf{X} + \sigma^2\mathbf{I}_m$ in a *reg*ularized *co*variance *m*atching (REGCOM) framework using the sample covariance matrix $\mathbf{C} = \frac{1}{T}\sum_{t=1}^T \mathbf{y}(t)\mathbf{y}^*(t)$. Let $\mathbf{c} \triangleq \text{vec}(\mathbf{C})$ and $\mathbf{r} \triangleq \text{vec}(\mathbf{X}) + \sigma^2\text{vec}(\mathbf{I}_m)$. The REGCOM estimator is given by solving

$$\min_{\mathbf{X}\in\mathcal{X}_S, \sigma^2} \left\|\mathbf{c} - \mathbf{r}(\mathbf{X}, \sigma^2)\right\|_{\mathbf{Q}}^2 + \lambda \left\|\mathbf{X}\right\|_*, \tag{4}$$

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where $\mathcal{X}_S$ is the class of matrices to which $\mathbf{X}$ belongs. For the problem considered in this work, $\mathcal{X}_S$ is a linear space of Hermitian Toeplitz matrices. Here $\mathbf{Q} \succ \mathbf{0}$ weights the residuals and λ penalizes the nuclear norm of the signal component. An appropriate value for the regularization parameter λ can be found from the data using the cross-validation method as we describe below. In short, the REGCOM estimator aims to find the 'sparsest' structured signal component $\mathbf{X}$ of the covariance matrix $\mathbf{R}$, consistent with the data $\mathbf{C}$.

A. Iteratively reweighted solution

We now present an approximate iterative solution to (4) drawing upon on [9]. First, note that a Hermitian Toeplitz matrix can be written as $\mathbf{X} = x_0\mathbf{S}_0 + \sum_{k=1}^{m-1}(x_k\mathbf{S}_k + x_k^*\mathbf{S}_k^\top)$, where

$$\mathbf{S}_k = \begin{bmatrix} \mathbf{0}_{m-k,k} & \mathbf{I}_{m-k} \\ \mathbf{0}_{k,m-k} & \mathbf{0}_{k,k} \end{bmatrix}, \quad k = 0, 1, \dots, m-1.$$

Then the linear structure can be parameterized as $\text{vec}(\mathbf{X}) = \mathbf{S}\boldsymbol{\theta}$, where $\boldsymbol{\theta} = [x_0 \ \text{Re}(x_1) \ \text{Im}(x_1) \ \cdots \ \text{Re}(x_{m-1}) \ \text{Im}(x_{m-1})] \in \mathbb{R}^{2m-1}$ and the structure matrix equals $\mathbf{S} = \boldsymbol{\Sigma}\boldsymbol{\Omega} \in \mathbb{C}^{m^2 \times (2m-1)}$, $\boldsymbol{\Sigma} = [\text{vec}(\mathbf{S}_0) \ \text{vec}(\mathbf{S}_1) \ \text{vec}(\mathbf{S}_1^\top) \cdots \ \text{vec}(\mathbf{S}_{N-1}) \ \text{vec}(\mathbf{S}_{N-1}^\top)]$ and

$$\boldsymbol{\Omega} = \begin{bmatrix} 1 & & & & & \\ & 1 & j & & & \\ & 1 & -j & & & \\ & & & \ddots & & \\ & & & & 1 & j \\ & & & & 1 & -j \end{bmatrix}$$

is a block-diagonal matrix [10].

Second, suppose $\mathbf{X}(\boldsymbol{\theta})$ is full-rank so that $\mathbf{W} = (\mathbf{X}\mathbf{X}^*)^{-1/2}$ exists. Using this weight matrix the nuclear norm can be written as $\|\mathbf{X}\|_* = \text{tr}\{\mathbf{W}\mathbf{X}\mathbf{X}^*\} = \text{tr}\{\mathbf{X}^*\mathbf{W}^{*/2}\mathbf{W}^{1/2}\mathbf{X}\}$ and so

$$\begin{aligned} \|\mathbf{X}\|_* &= \|\mathbf{W}^{1/2}\mathbf{X}\|_F^2 \\ &= \|\text{vec}(\mathbf{W}^{1/2}\mathbf{X})\|_2^2 \\ &= \|(\mathbf{I}_m \otimes \mathbf{W}^{1/2})\text{vec}(\mathbf{X})\|_2^2 \\ &= \left\| (\mathbf{I}_m \otimes \mathbf{W}^{1/2}) \begin{bmatrix} \mathbf{S} & \mathbf{0}_{m^2} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \sigma^2 \end{bmatrix} \right\|_2^2 . \end{aligned} \tag{5}$$

Define the effective parameters of $\mathbf{R}$ as $\boldsymbol{\eta} \triangleq [\boldsymbol{\theta}^\top \ \sigma^2]^\top$. Now we can recast problem (4) as

$$\hat{\boldsymbol{\eta}} = \underset{\boldsymbol{\eta} \in \mathbb{R}^{2m}}{\arg\min} \ \|\mathbf{c} - \mathbf{H}\boldsymbol{\eta}\|_{\mathbf{Q}}^2 + \lambda\|\mathbf{G}\boldsymbol{\eta}\|_2^2 \tag{6}$$

where $\mathbf{H} \triangleq [\mathbf{S} \ \text{vec}(\mathbf{I}_N)]$, $\mathbf{G} = (\mathbf{I}_m \otimes \mathbf{W}^{1/2})\widetilde{\mathbf{S}}$ and $\widetilde{\mathbf{S}} \triangleq [\mathbf{S} \ \mathbf{0}_{m^2}]$. Holding $\mathbf{W}$ fixed, the solution is

$$\hat{\boldsymbol{\eta}} = \left(\mathbf{H}^*\mathbf{Q}\mathbf{H} + \lambda\widetilde{\mathbf{S}}^*(\mathbf{I}_m \otimes \mathbf{W})\widetilde{\mathbf{S}}\right)^{-1}\mathbf{H}^*\mathbf{Q}\mathbf{c}. \tag{7}$$

Thus starting from an initial weight matrix, an approximate solution to the regularized covariance matching problem is given by updating $\hat{\boldsymbol{\eta}}$ and $\mathbf{W}$ in an alternating fashion.

If the Hermitian matrix $\mathbf{X}(\boldsymbol{\theta})$ were positive definite the weight matrix could be computed using the eigenvalue decomposition $\widehat{\mathbf{X}}(\hat{\boldsymbol{\theta}}) = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^*$ so that $\mathbf{W} = \mathbf{V}\boldsymbol{\Lambda}^{-1}\mathbf{V}^*$. The decomposition can be computed efficiently by exploiting the Hermitian Toeplitz structure, cf. [11]. As the estimator penalizes rank, however, positive definiteness is not ensured. The occurrence of indefinite solutions is remedied by retaining the smallest positive eigenvalue in $\boldsymbol{\Lambda}$, denoted $\lambda_i^\star > 0$, and setting any lower eigenvalues to $\lambda_i^\star/m$. Let $\widetilde{\boldsymbol{\Lambda}}$ denote the diagonal matrix of truncated eigenvalues, then the weight matrix is computed as $\mathbf{W} = \mathbf{V}\widetilde{\boldsymbol{\Lambda}}^{-1}\mathbf{V}^*$.

Further, for weighting the residuals $\mathbf{Q}$ can be chosen as $\mathbf{Q} = \mathbf{I}_{m^2}$. Alternatively we may update it iteratively using the estimated covariance matrix $\widehat{\mathbf{R}}(\hat{\boldsymbol{\eta}})$ from the previous iteration, i.e. $\mathbf{Q} = (\widehat{\mathbf{R}}^{-\top} \otimes \widehat{\mathbf{R}}^{-1})$, motivated by the covariance structure of the residuals for circular Gaussian statistics [12], [10].

The approximate iterative solution is summarized in Algorithm 1. Experimentally we find that when using few samples, the smallest eigenvalue of $\widehat{\mathbf{R}} = \widehat{\mathbf{X}} + \hat{\sigma}^2\mathbf{I}_m$ may occasionally become negative thereby making the covariance matrix indefinite. We address this by adding the corresponding magnitude to $\hat{\sigma}^2$. The weight matrices are initialized using the estimated received signal power, $\text{tr}\{\mathbf{C}\}$: $\mathbf{Q} = (\frac{m}{\text{tr}\{\mathbf{C}\}}\mathbf{I}_m \otimes \frac{m}{\text{tr}\{\mathbf{C}\}}\mathbf{I}_m)$ and $\mathbf{W} = 10^2 \times \frac{m}{\text{tr}\{\mathbf{C}\}}\mathbf{I}_m$ where the factor 10^2 is set to bias the initial estimate.

Algorithm 1 Approximate iterative solution

1: Input: $\mathbf{C}$, $\mathbf{S}$ and λ
2: Initialize $\mathbf{W}$ and $\mathbf{Q}$
3: $\mathbf{c} = \text{vec}(\mathbf{C})$ and $\widetilde{\mathbf{S}} = [\mathbf{S} \ \mathbf{0}_{m^2}]$
4: **repeat**
5: $\quad \hat{\boldsymbol{\eta}} := \left(\mathbf{H}^*\mathbf{Q}\mathbf{H} + \lambda\widetilde{\mathbf{S}}^*(\mathbf{I}_m \otimes \mathbf{W})\widetilde{\mathbf{S}}\right)^{-1}\mathbf{H}^*\mathbf{Q}\mathbf{c}$
6: $\quad [\mathbf{V}, \boldsymbol{\Lambda}] = \texttt{eig}\left(\text{mat}_{m,m}(\mathbf{S}\hat{\boldsymbol{\theta}})\right)$
7: $\quad$ Update $\mathbf{W}$ and $\mathbf{Q}$
8: **until** convergence
9: Output: $\widehat{\mathbf{R}} = \text{mat}_{m,m}(\mathbf{S}\hat{\boldsymbol{\theta}}) + \hat{\sigma}^2\mathbf{I}_m$

B. Selecting the regularization parameter

Finally we turn to the problem of inferring an appropriate value of the regularization parameter λ from the data $\mathbf{c}$ alone. Let $\mathbf{d} = \mathbf{Q}^{1/2}\mathbf{c}$ and $\widetilde{\mathbf{H}} = \mathbf{Q}^{1/2}\mathbf{H}$ so that we can re-write (7) compactly. The cross-validation method aims to predict the ith component d_i from the remaining components, denoted $\mathbf{d}^{(i)} \in \mathbb{C}^{m^2-1}$. We opt for the λ that minimizes the resulting prediction errors [13].

Let $\hat{\boldsymbol{\eta}}^{(i)}(\lambda)$ denote the estimate of $\boldsymbol{\eta}$ using $\mathbf{d}^{(i)}$ for a fixed λ and $\mathbf{h}_i^*$ be the ith row of $\widetilde{\mathbf{H}}$, then the weighted sum of squared prediction errors is

$$C(\lambda) = \frac{1}{m^2}\sum_{i=1}^{m^2} w_i \left| d_i - \mathbf{h}_i^*\hat{\boldsymbol{\eta}}^{(i)}(\lambda) \right|^2 . \tag{8}$$

Note that we can re-write (7) as $\hat{\boldsymbol{\eta}} = \mathbf{B}^{-1}(\lambda)\widetilde{\mathbf{H}}^*\mathbf{d}$, where $\mathbf{B}(\lambda) = (\widetilde{\mathbf{H}}^*\widetilde{\mathbf{H}} + \lambda\widetilde{\mathbf{S}}^*(\mathbf{I}_m \otimes \mathbf{W})\widetilde{\mathbf{S}})$ and where we have assumed the converged weight matrices $\mathbf{W}$ and $\mathbf{Q}$. Then the Hermitian matrix $\mathbf{\Gamma}(\lambda) \triangleq \mathbf{I}_{m^2} - \widetilde{\mathbf{H}}\mathbf{B}^{-1}(\lambda)\widetilde{\mathbf{H}}^*$ produces the residuals $\mathbf{e} = \mathbf{d} - \widetilde{\mathbf{H}}\hat{\boldsymbol{\eta}} = \mathbf{\Gamma}(\lambda)\mathbf{d}$. Following [14] we set the weights as $w_i \propto [\mathbf{\Gamma}(\lambda)]_{ii}^2 \geq 0$ which emphasizes the components d_i with large residual variances.

Next, the computation of the cost function $C(\lambda)$ can be simplified. Let $\widetilde{\mathbf{H}}_i \in \mathbb{C}^{(m-1)\times d}$ denote the observation matrix $\widetilde{\mathbf{H}}$ after removing the ith row. Then using the Sherman-Morrison formula we have

$$\begin{aligned}
\hat{\boldsymbol{\eta}}^{(i)} &= \left(\widetilde{\mathbf{H}}_i^*\widetilde{\mathbf{H}}_i + \lambda\widetilde{\mathbf{S}}^*(\mathbf{I}_m \otimes \mathbf{W})\widetilde{\mathbf{S}}\right)^{-1} \widetilde{\mathbf{H}}_i^*\mathbf{d}^{(i)} \\
&= (\mathbf{B} - \mathbf{h}_i\mathbf{h}_i^*)^{-1}\left(\widetilde{\mathbf{H}}^*\mathbf{d} - \mathbf{h}_i d_i\right) \\
&= \mathbf{B}^{-1}\left(\widetilde{\mathbf{H}}^*\mathbf{d} - \mathbf{h}_i d_i\right) + \frac{\mathbf{B}^{-1}\mathbf{h}_i\mathbf{h}_i^*\mathbf{B}^{-1}\left(\widetilde{\mathbf{H}}^*\mathbf{d} - \mathbf{h}_i d_i\right)}{1 - \mathbf{h}_i^*\mathbf{B}^{-1}\mathbf{h}_i} \\
&= \hat{\boldsymbol{\eta}} - \frac{\mathbf{B}^{-1}\mathbf{h}_i d_i - \mathbf{B}^{-1}\mathbf{h}_i\mathbf{h}_i^*\hat{\boldsymbol{\eta}}}{1 - \mathbf{h}_i^*\mathbf{B}^{-1}\mathbf{h}_i} \\
&= \hat{\boldsymbol{\eta}} - \frac{\mathbf{B}^{-1}\mathbf{h}_i}{1 - \mathbf{h}_i^*\mathbf{B}^{-1}\mathbf{h}_i}(d_i - \mathbf{h}_i^*\hat{\boldsymbol{\eta}}).
\end{aligned}$$

The prediction error can then be written as

$$\begin{aligned}
d_i - \mathbf{h}_i^*\hat{\boldsymbol{\eta}}^{(i)} &= d_i - \mathbf{h}_i^*\hat{\boldsymbol{\eta}} + \frac{\mathbf{h}_i^*\mathbf{B}^{-1}\mathbf{h}_i}{1 - \mathbf{h}_i^*\mathbf{B}^{-1}\mathbf{h}_i}(d_i - \mathbf{h}_i^*\hat{\boldsymbol{\eta}}) \\
&= \frac{1}{1 - \mathbf{h}_i^*\mathbf{B}^{-1}\mathbf{h}_i}(d_i - \mathbf{h}_i^*\hat{\boldsymbol{\eta}}) \\
&= \frac{1}{[\mathbf{\Gamma}(\lambda)]_{ii}}(d_i - \mathbf{h}_i^*\hat{\boldsymbol{\eta}}).
\end{aligned}$$

When the weights are normalized as $w_i = [\mathbf{\Gamma}(\lambda)]_{ii}^2/\mathrm{tr}\{\mathbf{\Gamma}(\lambda)\}^2$ [14], (8) can be written as

$$C(\lambda) = \frac{\|\mathbf{\Gamma}(\lambda)\mathbf{d}\|_2^2}{\mathrm{tr}\{\mathbf{\Gamma}(\lambda)\}^2}. \tag{9}$$

The approximate iterative solution to the regularized covariance matching problem then boils down to selecting the regularization parameter λ to minimize $C(\lambda)$ in (9).

IV. Experimental results

By means of simulation we compare the standard Capon beamformer using the sample covariance matrix (SCM) with the enhanced version using the regularized covariance matching estimate (REGCOM) in small sample and low-SNR conditions.

A. Setup

We consider a uniform linear array with $m = 10$ elements and half-wavelength spacing. Further, $d = 3$ uncorrelated sources are located at $\theta_1 = -15°$, $\theta_2 = 0°$ and $\theta_3 = 15°$. The waveforms are generated as white processes $s_i(t) \sim \mathcal{N}(0, p_i)$ with powers $p_1 = 2$, $p_2 = 3$ and $p_3 = 1$. We vary the signal to noise ratio SNR $\triangleq \mathrm{tr}\{\mathbf{P}/d\}/\mathrm{tr}\{\sigma^2\mathbf{I}_m/m\}$ and number of snapshots T. The statistical metrics are evaluated using 10^3 Monte Carlo simulations.

The regularization parameter λ is determined for each run using the cross-validation function $C(\lambda)$. Figure 1 shows an

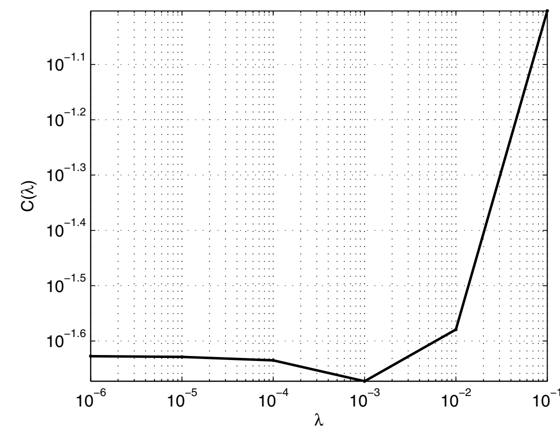

Fig. 1. Example realization of cross-validation function $C(\lambda)$.

example realization when sweeping λ. For the considered scenario we find that the minimum typically occurs in the region $[10^{-4}, 10^{-2}]$. For fast computation we evaluate $C(\lambda)$ at $\lambda \in \{10^{-4}, 10^{-3}, 10^{-2}\}$. In the process we obtain the corresponding covariance estimates.

B. Results

First we evaluate the estimates of $P(\theta)$ which provide valuable information about number of sources, their signal powers and directions of arrival. Figure 2 shows the mean of the estimates and their standard deviations for $T = 10$ snapshots. As can be seen REGCOM enhances the spectral estimates by significantly reducing the power bias associated with the Capon method. Unlike the standard method, three peaks are discernible with very high probability.

Next, we compare the spatial accuracy of the peaks of $\hat{P}(\theta)$. The DOAs are estimated by finding the maximizer of $\hat{P}(\theta)$ within the corresponding intervals. For θ_1 we search $\theta \in [-22.5°, -7.5°]$ using 1000 grid points. The performance is evaluated using the mean square error, MSE $= \mathrm{E}[|\theta_1 - \hat{\theta}_1|^2]$, as shown in Figure 3. For the small-sample regime there is a noticeable improvement across SNR -10 to 10 dB.

Finally, we evaluate the resulting linear filters $\mathbf{w}(\hat{\theta}_1)$ in estimating the waveform $s_1(t)$ arriving from θ_1. Figure 4 illustrates the normalized MSE of the waveform estimates $\hat{s}_1(t) = \mathbf{w}^*(\hat{\theta}_i)\mathbf{y}(t)$. The improvement for small samples is several decibel.

V. Conclusion

We formulated a regularized covariance matching framework using the nuclear norm, in order to exploit the unknown low-rank structure of the received signals for uniform linear arrays. An approximate iterative solution was developed along with a cross-validation method for inferring appropriate values of the regularization parameter from the data. The goal of the covariance matrix estimate was to enhance the nonparametric Capon method for small-sample scenarios. The simulation

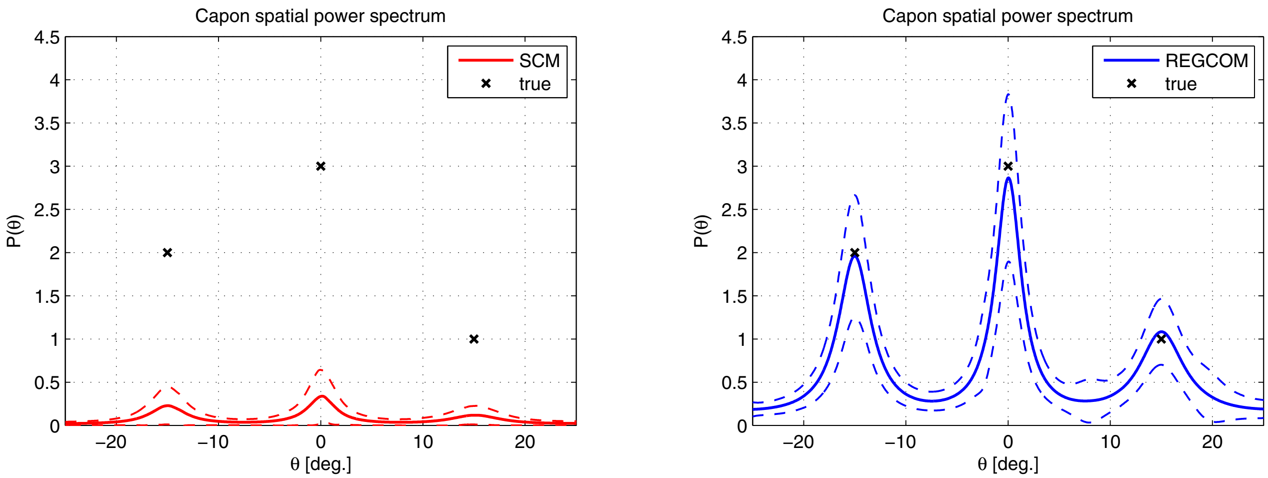

Fig. 2. $T = 10$ and SNR $= 0$ dB. Dotted lines indicate one standard deviation from mean.

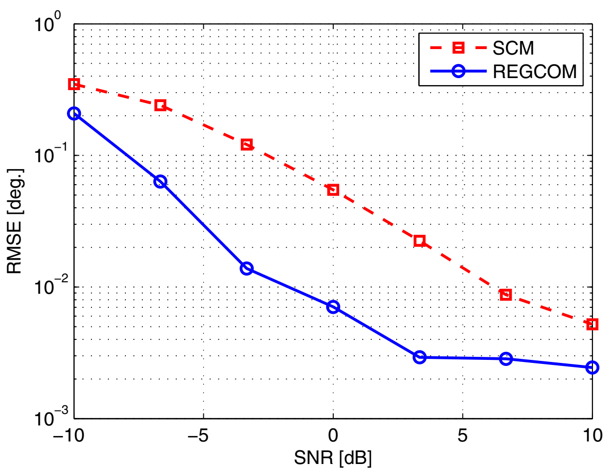

Fig. 3. Root MSE of direction of arrival estimate $\hat{\theta}_1$ versus SNR for $T = 10$.

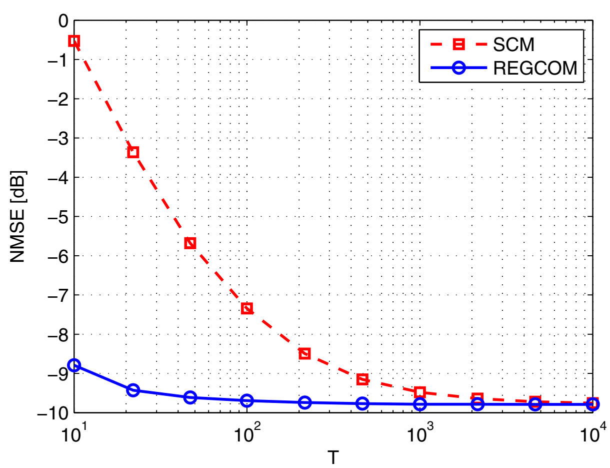

Fig. 4. Normalized MSE of waveform estimate $\hat{s}_1(t)$ versus snapshots T at SNR=0 dB.

results results show significantly improved spatial spectral and signal waveform estimates compared to the standard method.

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