Broadband Angle of Arrival Estimation Methods in a Polynomial Matrix Decomposition Framework

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***Abstract*—A large family of broadband angle of arrival estimation algorithms are based on the coherent signal subspace (CSS) method, whereby focussing matrices appropriately align covariance matrices across narrowband frequency bins. In this paper, we analyse an auto-focussing approach in the framework of polynomial covariance matrix decompositions, leading to comparisons to two recently proposed polynomial multiple signal classification (MUSIC) algorithms. The analysis is complemented with numerical simulations.**

I. Introduction

The coherent signal subspace (CSS) technique was proposed in [1] as an effective method for the estimation of angles of arrival of spectrally overlapping broadband sources using narrowband direction-finding algorithms such as MUSIC. A fundamental feature of CSS is that it coherently combines narrowband covariance matrices at different frequency bins covering the band occupied by the sources. In its simplest form, CSS pre-steers the array data by focussing matrices such that the sources of interest appear in the vicinity of the array's broadside, where array response vectors for all temporal frequencies approximately coincide. The focussing matrices can be constructed from approximate knowledge of the signals of interest's directions of arrival (DoA).

The long evolution of CSS algorithms since this inception, see e.g. [2], [3], includes a recently proposed auto-focussing approach [5]. In this method, focussing matrices are directly calculated from the array's space-time covariance matrix without the requirement for explicit knowledge of approximate DoAs.

Different from CSS, where the wideband approach is cleverly bypassed in favour of narrowband processing, an EVD algorithm for polynomial space-time covariance matrices [8] has recently led to a broadband MUSIC algorithm [7]. Applicable directly to broadband array data, these polynomial MUSIC algorithms exploit the idea of signal subspaces created by the polynomial EVD [8]. While this approach seems distinct from CSS, the purpose of this paper is to highlight the similarities by expressing the auto-focussing approach [5] in terms of polynomial matrix decompositions.

Below, we define a broadband steering vector, the polynomial space-time covariance matrix and polynomial EVD in Sec. II, and review the auto-focussing broadband approach in Sec. III. The formulation of MUSIC based on the auto-focussing method of [5] is then related to the two polynomial MUSIC algorithms in [7] in Sec. IV. Two illustrative simulations are included in Sec. V, followed by conclusions in Sec. VI.

II. System Model

Based on the signal model for a broadband array described in Sec. II-A, Sec. II-B defines a polynomial space-time covariance matrix.

A. Broadband Steering Vector

An M-element array of omnidirectional sensors located at positions $\mathbf{r}_m$, $m = 1 \dots M$ collects broadband data in a vector $\mathbf{x}[n] \in \mathbb{C}^M$. For the lth far-field source $s_l[n]$, the array experiences a planar wavefront with normal $\mathbf{k}_l$. We are only interested in the relative delay between signals at the sensors, such that the contributions to $\mathbf{x}[n]$ in the absence of attenuation due to propagation is

$$\mathbf{x}[n] = \sum_l \sum_{\nu=0}^{\infty} \mathbf{a}_l[\nu] s_l[n-\nu] \ + \ \mathbf{v}[n] \tag{1}$$

with the broadband steering vector $\mathbf{a}_i[n]$

$$\mathbf{a}_l[n] = \frac{1}{\sqrt{M}} \begin{bmatrix} \delta[n-\tau_{l,0}] \\ \vdots \\ \delta[n-\tau_{l,M-1}] \end{bmatrix} \tag{2}$$

and the normalised delays $\tau_{l,m} = \frac{\mathbf{k}_l^{\mathrm{H}} \mathbf{r}_m}{c T_{\mathrm{s}}}$, whereby T_{s} is the sampling period and c the propagation speed in the medium, such that $\mathbf{k}_l/c$ is the lth source's slowness vector. The vector $\mathbf{v}[n]$ adds spatially and temporally uncorrelated noise with covariance $\mathcal{E}\{\mathbf{v}[n]\mathbf{v}^{\mathrm{H}}[n]\} = \sigma_v^2 \mathbf{I}$ to the model in (1). Below, $\mathbf{a}_{\vartheta,\varphi}[n]$ refers to a broadband steering vector determined by $\mathbf{k}$ with azimuth φ and elevation ϑ.

B. Space-Time Covariance Matrix and Polynomial EVD

Collecting an M-element broadband array data vector $\mathbf{x}[n] \in \mathbb{C}^M$, its space-time covariance matrix is given by

$$\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\} \quad , \tag{3}$$

which forms a transform pair with the cross power spectral density (CSD) matrix $\mathbf{R}(z) \bullet\!\!-\!\!\circ\, \mathbf{R}[\tau]$,

$$\mathbf{R}(z) = \sum_\tau \mathbf{R}[\tau] z^{-\tau} \quad . \tag{4}$$

The CSD matrix is parahermitian, i.e. $\mathbf{R}(z) = \tilde{\mathbf{R}}(z) = \mathbf{R}^{\mathrm{H}}(1/z^*)$. Based on (1) and with $\mathbf{a}_l[n] \circ\!\!-\!\!\bullet\, \mathbf{a}_l(z)$, it can also be expressed as

$$\mathbf{R}(z) = \sum_l \mathbf{a}_l(z) S_l(z) \tilde{\mathbf{a}}_l(z) \; + \; \sigma_v^2 \mathbf{I} \quad , \tag{5}$$

with $S_l(z)|_{z=e^{j\Omega}}$ the power spectral density (PSD) of the lth source signal, $s_l[n]$.

A polynomial EVD [8] decouples the parahermitian $\mathbf{R}(z)$ by means of a paraunitary $\mathbf{Q}(z)$,

$$\mathbf{\Gamma}(z) = \tilde{\mathbf{Q}}(z)\mathbf{R}(z)\mathbf{Q}(z) \quad , \tag{6}$$

such that $\mathbf{\Gamma}(z) = \mathrm{diag}\{\Gamma_1(z) \;\; \Gamma_1(z) \;\; \ldots \;\; \Gamma_M(z)\}$ is diagonalised and spectrally majorised with PSDs $\Gamma_{i+1}(e^{j\Omega}) \geq \Gamma_i(e^{j\Omega}) \quad \forall \Omega, \quad i = 1 \ldots (M-1)$, with $\Gamma_i(e^{j\Omega}) = \Gamma_i(z)|_{z=e^{j\Omega}}$. Below, we use this decomposition framework to express the CSS approach.

III. Coherent Covariance and Auto-Focussing Matrices

A. Coherent Signal Subspace Method

Based on a K-point DFT of the space-time covariance matrix,

$$\mathbf{R}(e^{j\Omega_k}) = \sum_{\tau=0}^{K-1} \mathbf{R}[\tau] e^{-j\Omega_k \tau} \tag{7}$$

with $\Omega_k = \frac{2\pi}{K}k$, $k = 0 \ldots K-1$, the CSS method is based on a covariance matrix

$$\mathbf{R}_{\mathrm{coh}} = \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{T}(e^{j\Omega_k}) \mathbf{R}(e^{j\Omega_k}) \mathbf{T}^{\mathrm{H}}(e^{j\Omega_k}) \quad , \tag{8}$$

obtained by coherently combining across frequency bins through unitary and frequency-dependent focussing matrices $\mathbf{T}(e^{j\Omega})$.

Following the "auto-focussing" approach of [5], for a reference frequency Ω_0, an EVD of the appropriate frequency-bin covariance matrix $\mathbf{R}(e^{j\Omega_0})$ yields

$$\mathbf{\Lambda}_o = \mathbf{Q}_0^{\mathrm{H}} \mathbf{R}(e^{j\Omega_0}) \mathbf{Q}_0 \quad . \tag{9}$$

Together with the modal matrix $\mathbf{Q}^{\mathrm{H}}(e^{j\Omega_k})$ extracted for frequency bin k, $k = 0 \ldots (K-1)$, the auto-focussing matrix is constructed as

$$\mathbf{T}(e^{j\Omega_k}) = \mathbf{Q}_0 \mathbf{Q}^{\mathrm{H}}(e^{j\Omega_k}) \quad . \tag{10}$$

Therefore, the coherent covariance matrix in (8) can be diagonalised by $\mathbf{Q}_0$ to provide

$$\mathbf{\Lambda}_{\mathrm{coh}} = \mathbf{Q}_0^{\mathrm{H}} \mathbf{R}_{\mathrm{coh}} \mathbf{Q}_0 = \mathrm{diag}\{\lambda_1 \;\; \lambda_2 \;\; \ldots \;\; \lambda_M\} \quad , \tag{11}$$

with λ_m, $m = 1 \ldots M$ the eigenvalues of $\mathbf{R}_{\mathrm{coh}}$ in (8).

B. Auto-Focusssing Approach via CSD Matrix and PEVD

With the modal matrix $\mathbf{Q}_0$ obtained at the reference frequency via EVD of (7), the focussing matrix can be formulated as a paraunitary matrix $\mathbf{T}(z)|_{z=e^{j\Omega}} = \mathbf{Q}_0 \mathbf{Q}^{\mathrm{H}}(e^{j\Omega})$. Replacing the summation over frequency bins in (8) by the integration over the Fourier transform (i.e. $K \to \infty$) leads to

$$\mathbf{R}_{\mathrm{coh}} \approx \frac{1}{2\pi} \oint \left\{ \mathbf{T}(z)\mathbf{R}(z)\tilde{\mathbf{T}}(z) \right\}_{z=e^{j\Omega}} d\Omega \tag{12}$$

$$= \mathbf{Q}_0 \frac{1}{2\pi} \oint \left\{ \tilde{\mathbf{Q}}(z)\mathbf{R}(z)\mathbf{Q}(z) \right\}_{z=e^{j\Omega}} d\Omega \; \mathbf{Q}_0^{\mathrm{H}} \, . \tag{13}$$

Since the paraunitary matrix $\mathbf{Q}(z)$ diagonalises $\mathbf{R}(z)$, the argument under the integral is the polynomial EVD of (6), resulting in a diagonal matrix of power spectral densities, $\frac{1}{2\pi}\oint \mathbf{\Gamma}(e^{j\Omega})d\Omega = \mathbf{\Gamma}[0]$, where $\mathbf{\Gamma}[0]$ is the evaluation of $\mathbf{\Gamma}[\tau] \circ\!\!-\!\!\bullet\, \mathbf{\Gamma}(z)$ for zero lag. Therefore

$$\mathbf{R}_{\mathrm{coh}} \approx \mathbf{Q}_0 \mathbf{\Gamma}[0] \mathbf{Q}_0^{\mathrm{H}} = \mathbf{Q}_0 \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_M^2 \end{bmatrix} \mathbf{Q}_0^{\mathrm{H}} \tag{14}$$

represents the coherent covariance matrix in terms of the polynomial EVD of the CSD matrix.

Given that the DFT in (7) is a sufficiently accurate representation of the Fourier transform formulation in (12), then (11) and (14) are equivalent with $\mathbf{\Lambda} = \mathbf{\Gamma}[0]$. Further, the PEVD of the CSD matrix provides a paraunitary $\mathbf{Q}(z)$ that leads to an auto-focussing matrix $\mathbf{Q}_0\tilde{\mathbf{Q}}(z)$ that is continuous in frequency.

IV. Broadband Angle of Arrival Estimation

A. MUSIC Based on Auto-Focussing Approach

Based on the coherent covariance matrix, the standard MUSIC algorithm can be applied by probing the noise-only subspace of $\mathbf{R}_{\mathrm{coh}}$ with a set of steering vectors at the reference frequency Ω_0.

If the eigenvalues $\mathbf{R}_{\mathrm{coh}}$ reveal R linearly independent sources, then the last $M-R$ columns of $\mathbf{Q}_0 = \left[\mathbf{Q}_{0,\mathrm{s}} \mathbf{Q}_{0,\mathrm{s}}^{\perp}\right]$ contained in $\mathbf{Q}_{0,\mathrm{s}}^{\perp} \in \mathbb{C}^{M \times (M-R)}$ span the noise-only subspace of the coherent covariance matrix. Scanning for azimuth and elevation angles, $\mathbf{a}_{\varphi,\vartheta}(z)$ can be evaluated at the reference frequency Ω_0, leading to the MUSIC spectrum

$$S_{\mathrm{AF}}(\varphi,\vartheta) = \|\mathbf{Q}_{0,\mathrm{s}}^{\perp} \mathbf{a}_{\varphi,\vartheta}(e^{j\Omega_0})\|_2^{-2}$$
$$= \frac{1}{\mathbf{a}_{\varphi,\vartheta}^{\mathrm{H}}(e^{j\Omega_0}) \mathbf{Q}_{0,\mathrm{s}}^{\perp,\mathrm{H}} \mathbf{Q}_{0,\mathrm{s}}^{\perp} \mathbf{a}_{\varphi,\vartheta}(e^{j\Omega_0})} \quad . \tag{15}$$

B. Polynomial Spatio-Spectral MUSIC

For the polynomial MUSIC algorithm, a spatial and a spatio-spectral version have been suggested in [7]. Both versions require the identification of the polynomial noise-only subspace given R sources identified from $\mathbf{\Gamma}(z)$,

$$\mathbf{Q}(z) = \left[\mathbf{Q}_{\mathrm{s}}(z) \mathbf{Q}_{\mathrm{s}}^{\perp}(z)\right] \quad , \tag{16}$$

where $\mathbf{Q}_\mathrm{s}^\perp \in \mathbb{C}^{M\times R}(z)$. The polynomial spatio-spectral (PSS) MUSIC is based on inverting a PSD-type function,

$$S_\mathrm{PSS}(\varphi,\vartheta,e^{j\Omega}) = \frac{1}{\tilde{\mathbf{a}}_{\varphi,\vartheta}(z)\tilde{\mathbf{Q}}_\mathrm{s}^\perp(z)\mathbf{Q}_\mathrm{s}^\perp(z)\mathbf{a}_{\varphi,\vartheta}(z)}\Big|_{z=e^{j\Omega}} \quad . \tag{17}$$

Provided that the estimation of the number of linearly independent sources, R, is the same from (11) for auto-focussing (AF) and from (14) of the polynomial approach, then with $\mathbf{Q}_0$ being the evaluation of the paraunitary $\mathbf{Q}(z)$ at the reference frequency Ω_0, i.e. $\mathbf{Q}_0 = \mathbf{Q}(z)|_{z=e^{j\Omega_0}}$, it follows that

$$S_\mathrm{AF}(\varphi,\vartheta) = S_\mathrm{PSS}(\varphi,\vartheta,e^{j\Omega})|_{\Omega=\Omega_0} \quad . \tag{18}$$

Therefore, the auto-focussing approach to coherent signal subspace MUSIC estimation is equivalent to evaluating the polynomial spatio-spectral MUSIC spectrum at the reference frequency Ω_0.

To obtain the same spatio-spectral characterisation of the array data as provided by PSS-MUSIC with the auto-focussing approach, a sequence of different modal matrices $\mathbf{Q}_0$ at different reference frequencies Ω_0 could be calculated, for all of which (15) is evaluated.

C. Polynomial Spatial MUSIC

The polynomial spatial (PS) MUSIC estimate [7] integrates the PSD in the denominator of (17), providing a power term

$$\gamma = \frac{1}{2\pi}\oint \left(\tilde{\mathbf{a}}_{\varphi,\vartheta}(z)\tilde{\mathbf{Q}}_\mathrm{s}^\perp(z)\mathbf{Q}_\mathrm{s}^\perp(z)\mathbf{a}_{\varphi,\vartheta}(z)\right)\Big|_{z=e^{j\Omega}} d\Omega\,. \tag{19}$$

The PS-MUSIC spectrum is given by the reciprocal of (19),

$$S_\mathrm{PS}(\varphi,\vartheta) = \frac{1}{\gamma}\,. \tag{20}$$

If the integral in (19) is approximated by a sum over discrete frequency bins, i.e.

$$\gamma \approx \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{a}_{\varphi,\vartheta}^\mathrm{H}(e^{j\Omega_k})\mathbf{Q}_\mathrm{s}^{\perp,\mathrm{H}}(e^{\Omega_k})\mathbf{Q}_\mathrm{s}^\perp(e^{j\Omega_k})\mathbf{a}_{\varphi,\vartheta}(e^{j\Omega_k}), \tag{21}$$

then (21) is the summation over the denominator terms of (15) for all possible reference frequencies Ω_k with $\Omega_k = \frac{2\pi}{K}k$, $k = 0\dots(K-1)$. The paraunitary matrix $\mathbf{Q}(z)$ that feeds into (21) has been demonstrated in (14) to cohere the spatio-temporal covariance matrixin the auto-focussing sense.

V. Numerical Simulations

A. Implementational Aspects

It has been shown in [9] that the polynomial EVD in (6) fulfilling spectral majorisation can be approximated very closely by FIR paraunitary matrices even if an exact decomposition by FIR filter banks does not exist. Therefore, here we rely on the second order sequential best rotation (SBR2) algorithm [8], which iteratively approaches the decomposition in (6), and has been proven to converge, whereby the number of iterations will determine the accuracy with which diagonalisation and spectral majorisation are approximated.

Broadband steering vectors are based on fractional delay filters constructed from truncated sinc functions, which can be substantially improved by applying a tapered window [10].

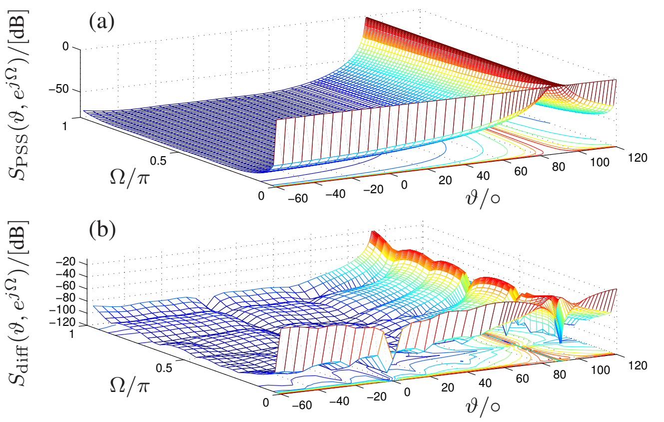

Fig. 1. (a) PSS-MUSIC spectrum and (b) difference to AF-MUSIC for a single source at $\vartheta = 90°$.

B. Idealistic Example with Exact PEVD

As a simple toy problem, a single source emits an uncorrelated Gaussian signal. In an otherwise noise-free scenario, this signal is received by an $M = 4$ element, spatially and temporally critically sampled array from endfire position, such that the broadband steering vector of the source is

$$\mathbf{a}_1(z) = \frac{1}{\sqrt{M}}\left[1 \;\; z^{-1} \;\; \dots \;\; z^{-M+1}\right]^\mathrm{T} \tag{22}$$

providing a space-time covariance matrix

$$\mathbf{R}_1(z) = \begin{bmatrix} 1 & z^1 & \dots & z^{M-1} \\ z^{-1} & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ z^{-M+1} & \dots & \dots & 1 \end{bmatrix} \quad . \tag{23}$$

Because $\mathbf{R}_1(z)$ is rank one, a manifold of diagonalising decompositions exists, with one possibility

$$\mathbf{Q}(z) = \mathrm{diag}\left\{1 \; z^{-1} \; \dots \; z^{-M+1}\right\}\mathbf{T}_\mathrm{DFT}, \tag{24}$$

where $\mathbf{T}_\mathrm{DFT}$ is a normalised, unitary M-point DFT matrix. For PSS-MUSIC in (17), the spectrum in Fig. 1(a) emerges, identifying the DoA of the end-fire source. In line with broadband arrays, at lower frequencies the fixed aperture degrades the spatial resolution, with no ability to discern sources at DC.

For auto-focussing, at a given reference frequency Ω_0, it can be shown that $\mathbf{R}_{\mathrm{coh},\Omega_0} = \mathbf{R}(z)|_{z=e^{j\Omega_0}}$ and $\mathbf{\Lambda}_{\mathrm{coh},\Omega_0} = \mathrm{diag}\{1, 0\cdots 0\}$. Using the nullspace $\mathbf{Q}_\mathrm{s}^\perp(e^{j\Omega_0})$ derived from the EVD of $\mathbf{R}_{\mathrm{coh},\Omega_0}$, the MUSIC spectrum is evaluated for a range of $K = 64$ reference frequencies Ω_0. This leads to a spectrum very closely related to PSS-MUSIC, with the difference, $S_\mathrm{diff}(\vartheta,e^{j\Omega}) = |S_\mathrm{PSS}(\vartheta,e^{j\Omega}) - S_\mathrm{AF}(\vartheta,\Omega)|$, plotted in Fig.1(b). The error reaches a maximum of 10dB where the PSS-MUSIC spectrum is numerically most sensitive, i.e. towards the source at $\vartheta = 90°$, and for DC, $\Omega = 0$, which can be attributed to the inaccuracies in implementing broadband steering vector. Note the trivial broadband steering vector towards broadside $\vartheta = 0°$, $\mathbf{a}_{0°}(z) = \frac{1}{M}[1\;1\;\dots]^\mathrm{T}$, for which the error in Fig. 1(b) is neglegible.

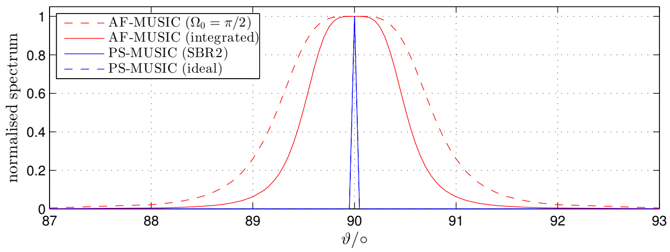

Fig. 2. Comparison between AF-MUSIC (i) at $\Omega_0 = \frac{\pi}{2}$, (ii) integrated according to (21), and PS-MUSIC (iii) with estimated and (iv) ideal PEVD.

Due to the trivial space-time covariance matrix, SBR2 converges instantly to the exact PEVD, yielding exact results for PS-MUSIC in Fig. 2. AF-MUSIC is shown both for a single reference frequency, and integrated over a range of reference frequencies, with results somewhat degraded compared to PS-MUSIC.

C. Realistic Scenario

For a more realistic scenario, we consider an $M = 8$ element array illuminated by a mixture of three mutually uncorrelated Gaussian sources of equal power,

- $\vartheta_1 = -30°$, active over range $\Omega \in [\frac{3\pi}{8}; \pi]$,
- $\vartheta_2 = 40°$, active over range $\Omega \in [\frac{\pi}{2}; \pi]$, and
- $\vartheta_3 = 20°$, active over range $\Omega \in [\frac{2\pi}{8}; \frac{7\pi}{8}]$,

mixed with uncorrelated Gaussian noise 30dB below the three directional signals. Fig. 3 shows the PSS- and CSS-MUSIC spectra, in the latter case evaluated for a set of reference frequencies over small fractional bandwidths. In spectral ranges where all sources are active, AF-MUSIC provides superior resolution, also indicated by a snapshot for $\Omega = \frac{3\pi}{4}$ in Fig. 4; outside the overlap region, the performance is degraded. PS-MUSIC in Fig. 4 provides a lower resolution than auto-focussing, but is calculated over the entire spectrum, hence not requiring any prior spectral knowledge.

VI. Conclusions

Polynomial MUSIC algorithms have been compared to a recently proposed auto-focussing (AF) approach, which is claimed to be in the line of coherent signal subspace methods [5]. With the AF approach expressed in the framework of polynomial space-time covariance matrices and their polynomial eigenvalue decomposition, and under the assumption the DFT sufficiently well approximating the Fourier transform, the polynomial spatio-spectral MUSIC algorithm has been shown to equate to the AF approach when evaluated at the reference frequency, while the polynomial spatial MUSIC algorithm has been shown to relate to a summation of AF terms for a set of reference frequencies.

Numerical simulations have indicated that the polynomial MUSIC methods perform similar to the AF-approach where the exact PEVD is known or easily determined. For more realistic scenarios, restricting AF to sensible fractional bandwidths will provide superior resolution over polynomial MUSIC; however, the latter does not rely on a-priori spectral information and can be calculated over the entire bandwidth with appealing results.

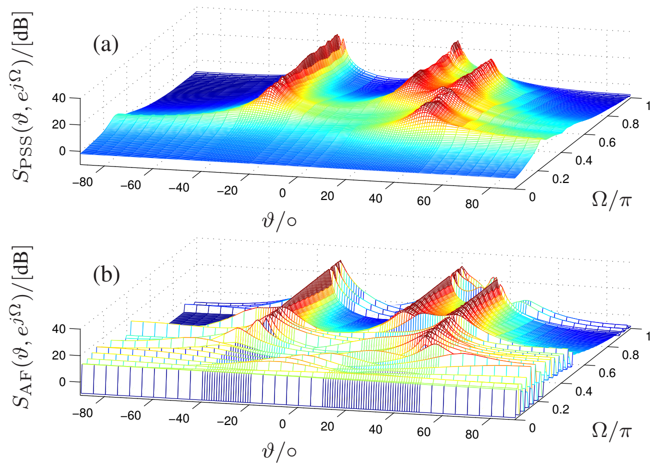

Fig. 3. (a) PSS- and (b) AF-MUSIC spectra for 3 overlapping sources.

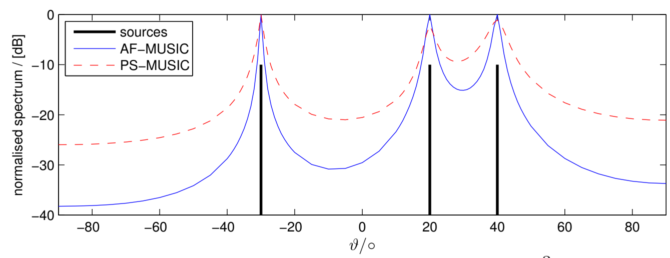

Fig. 4. PS- and AF-MUSIC (measured at $\Omega_0 = \frac{3}{4}\pi$)

VII. Acknowledgement

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References

[1] H. Wang and M. Kaveh, "Coherent signal-subspace processing for the detection and estimation of angles of arrival of multiple wide-band sources," *IEEE Trans ASSP*, **33**(4):823–831, Aug. 1985.

[2] H. Wang and M. Kaveh, S "On the performance of signal-subspace processing–part ii: Coherent wide-band systems," *IEEE Trans ASSP*, **35**(11):1583–1591, Nov. 1987.

[3] H. Hung and M. Kaveh, "Focussing matrices for coherent signal-subspace processing," *IEEE Trans ASSP*, **36**(8):1272–1281, Aug. 1988.

[4] M.A. Doron and A.J. Weiss, "On focusing matrices for wide-band array processing," *IEEE Trans SP*, **40**(6):1295–1302, June 1992.

[5] P. Pal and P.P. Vaidyanathan, "A novel autofocusing approach for estimating directions-of-arrival of wideband signals," in *43rd Asilomar Conf. Signals, Systems & Computers*, Pacific Grove, CA, pp. 1663–1667, Nov. 2009.

[6] M.A. Doron, E. Doron, and A.J. Weiss, "Coherent wide-band processing for arbitrary array geometry," *IEEE Trans SP*, **41**(1):414, Jan. 1993.

[7] M. Alrmah, S. Weiss, and S. Lambotharan, "An extension of the music algorithm to broadband scenarios using polynomial eigenvalue decomposition," in *19th European Signal Processing Conf.*, Barcelona, Spain, pp. 629–633, Aug. 2011.

[8] J.G. McWhirter, P.D. Baxter, T. Cooper, S. Redif, and J. Foster, "An EVD Algorithm for Para-Hermitian Polynomial Matrices," *IEEE Trans SP*, **55**(5):2158–2169, May 2007.

[9] S. Icart and P. Comon, "Some properties of Laurent polynomial matrices," in *9th IMA Conf. Mathematics in Signal Processing*, Birmingham, UK, Dec. 2012.

[10] J. Selva, "An efficient structure for the design of variable fractional delay filters based on the windowing method," *IEEE Trans SP*, **56**(8):3770–3775, Aug. 2008.