Error Exponents for Bias Detection of a Correlated Process over a MAC Fading Channel

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Abstract—In this paper, we analyze a binary hypothesis testing problem using a wireless sensor network (WSN). Using Large Deviation Theory (LDT), we compute the exponents of the error probabilities for the detection of a constant under a correlated process. Each sensor transmits its local measurement through a multiple-access (MAC) Rician fading channel with a line-of-sight (LOS) component to the fusion center (FC) using an uncoded analog scheme. The FC decides if the constant is present or not. We examine the behavior of the error exponents as a function of the correlation process and the fading LOS component. We also show that this scheme achieves the centralized error exponents when the number of sensors approaches infinity even when the fading LOS paths between the sensors and the FC are not so strong and the underlaying process is correlated. In this way, neither feedback between the FC and the sensors nor cooperation between the sensors is necessary to provide a sufficient statistic to the FC.

I. INTRODUCTION

In this work, we analyze the performance of a detection problem with two hypotheses in the context of large wireless sensor networks (WSN). Spatially distributed sensors take measurements and communicate them to a fusion center (FC), where the final decision is made. In particular, we investigate the distributed detection problem of deciding if the bias of a correlated process is present or not. For simplicity of exposition we assume that the bias is constant. The extension to any deterministic signal is straightforward [1].

Usually a large WSN is built of several hundreds or even thousands nodes with the ability to sense some physical magnitude. As sensors are packed closer, it is reasonable to expect that their measurements become more correlated [2]. A good measure of the performance of large networks are the error exponents as the amount of sensors goes to infinity. The error exponents give an estimate of the number of sensors required to reach a certain error probability.

We assume that the sensors use an uncoded analog scheme to transmit their local measurements [3]–[5]. Several works on distributed detection have been done assuming that sensors communicate with the FC through orthogonal channels [2], [6], [7]. However, for a large WSN, this assumption implies a large bandwidth requirement for simultaneous transmission or a large detection delay. On the other hand, a much more bandwidth-efficient usage of the channel is to use a multipleaccess channel (MAC), where the sensors transmit simultaneously on the same bandwidth. Due to the nature of the wireless channel, the FC receives the superposition of all sensor measurements. The key point is to design the transmit signal in such a way that the received signal at the FC becomes a sufficient statistic. The case of a Gaussian MAC channel without fading with independent and identically distributed (i.i.d.) observations was considered in [8]. There, it is shown that the analog transmission of the log-likelihood ratio (LLR) asymptotically achieves the centralized error exponent, when the number of sensor approaches infinity. In this paper, we calculate the error exponents in the Neyman-Pearson framework for a Gaussian MAC Rician fading channel [9] using large deviation theory (LDT) and show its dependency with the process correlation and with the fading characteristics. In the limit, when the variance of the fading vanishes, we recover the result shown in [1]. Also, we will work with a MAC channel under perfect synchronization. The case in which some phase uncertainty exists in the communication link was analyzed in [1].

The paper is organized as follows. In Section II, we present the network model. In Section III, we establish the extended Toeplitz and Gärtner-Ellis theorems used in the derivation of the error exponents calculated in Section IV. Some numerical results are shown in Section V and the main conclusions are mentioned in Section VI.

II. WIRELESS SENSOR NETWORK MODEL

We consider a binary hypothesis testing problem where each one of the n sensors obtains the following the measurements under both hypotheses

$$\begin{cases} \mathcal{H}_1: & x_k = \theta_0 + z_k, \\ \mathcal{H}_0: & x_k = z_k, \end{cases} \quad k = 1, \dots, n.$$

$$(1)$$

Here, $\theta_0 \in \mathbb{R}$ is the bias or constant¹ to be detected and $z_n = [z_1, \ldots, z_n]^T$ is a Gaussian real-valued correlated vector with zero-mean and Toeplitz covariance matrix $\Sigma^{(n)}$. We also define the measurement vector $x_n = [x_1, \ldots, x_n]^T$ that takes the form $x_n = \theta_0 \mathbf{1}_n + z_n$ under \mathcal{H}_1 and $x_n = z_n$ under \mathcal{H}_0 , where $\mathbf{1}_n$ is an *n*-dimensional vector of ones.

In the past years, several schemes were presented in the literature where the log-likelihood ratio (LLR) is transmitted

This work was partially supported by the Peruilh grant of the UBA and project UBACYT 2002010200250.

¹This model and thereafter results can be easily extended to a deterministic signal as in [1].

to the FC [10]–[12]. Note that in the case of (1), the local LLR's are affine function of the measurements. We assume that all sensors have the same limited average power constraint and that they transmit their local measurements to the fusion center (FC) using an analog uncoded scheme trough a multiple-access fading channel². The signal received at the FC is

$$\tilde{y}_n = \sum_{k=1}^n \tilde{h}_k x_k + \tilde{w} \tag{2}$$

where \tilde{w} is an additive circular complex Gaussian noise with zero-mean and variance $\sigma_{\tilde{w}}^2$ and \tilde{h}_k takes into account the *k*-th fading channel and the phase uncertainties present in the communication links between the sensors and the FC due to the different lengths of the wireless paths and the fact that each node usually use independent local oscillators.

As pointed out in [13], these kind of schemes do not work under zero-mean channels. In this paper, we assume that the sensor network and the FC are fixed and the fading channels follow a complex Gaussian distribution with approximately equal line-of-sight (LOS) electrical paths. This is a common and practical hypothesis given that the FC can be deployed with an antenna in a high position with respect to the sensors. Nonetheless, each of the LOS paths between each sensor and the FC, can present a different phase, due mainly to different physical characteristics in each LOS path and different mismatches between the oscillators at the nodes and the FC. This could lead to destructive interferences between the LOS paths of the *n* sensors as typically occurs in zero-mean channels. In order to avoid this it is necessary to compensate each mean channel phase at the sensors (transmitters) to have a MAC channel with non-zero mean and constructive interference between the LOS paths.

The channel phase compensation does not increase the transmit power at the sensors and can be done easily in practice by using the reverse channel (from the FC to the sensors) in a time division fashion (TDD). The compensation scheme can be done as follows. The FC broadcasts a pilot sequence and each sensor compares the received signal with its local reference and obtains a phase estimation. The duration of the pilot signal T_p must be long enough with respect to the coherence time of the fading channel such that the ergodicity of the channel mean (which it is assumed to remain fixed during period T_p). This is a one time procedure that has to be performed at the beginning of the network operation. Depending on the thermal drift of the oscillators, the speed at which the environment changes, etc, an update procedure should be necessary at a regular basis.

Finally, the compensated or equivalent channels can be represented by *n* complex Gaussian random variables with mean $\mu_{\tilde{h}} = |\mu_{\tilde{h}}|e^{j\psi}$ and variance $\sigma_{\tilde{h}}^2$, where ψ is the common reference phase recovered after the estimation procedure described above. Without loss of generality, we assume that $\psi = 0$, and $|\mu_{\tilde{h}}| = \mu_h > 0$. The signal received at the FC is a complex random variable and both, quadrature and in-phase components, have information sent by the sensors. However, both components present an important difference: the quadrature component channel is zero-mean and it only

carries information of the energy of the sensed signal while the in-phase channel has a non-zero-mean. As it is shown in [13] in the non-fading case, there is no loss of asymptotic optimality (when the number of sensors approaches infinity) if the quadrature component is discarded at the FC because its mean under both hypothesis is null. Therefore, we only consider the real or in-phase component of the received signal at the FC,

$$y_n = \frac{1}{n} \sum_{k=1}^n h_k x_k + \frac{w}{n},$$
 (3)

where $h_k \sim \mathcal{N}(\mu_h, \sigma_h^2)$, $w \sim \mathcal{N}(0, \sigma_w^2)$, with $\sigma_h^2 = \sigma_{\tilde{h}}^2/2$, $\sigma_w^2 = \sigma_{\tilde{w}}^2/2$, and we have normalized the received signal by the number of sensors. The test performed by the FC is the following: Choose \mathcal{H}_1 if $y_n > \tau$, and choose \mathcal{H}_0 otherwise, where τ is the predefined threshold of the test.

III. PRELIMINARY TOOLS

In this section, we introduce the main tools to compute the error exponents. First, an extension of the Toeplitz distribution theorem is presented and then, the Gärtner-Ellis theorem is enunciated.

Theorem 1 (Extended Toeplitz Distribution [14]): Let $\{s_k\}_{k=1}^n$ be a deterministic signal with spectral density $P(\omega)$ and define the vector $s_n = [s_1, \ldots, s_n]^T$. For an absolutely summable Toeplitz matrix $\Sigma^{(n)}$ with spectral density $S(\omega)$, let $\{\lambda_k^{(n)}\}_{k=1}^n$ be the eigenvalues of $\Sigma^{(n)}$ contained on the interval $[\delta_1, \delta_2]$, and $\{\phi_k^{(n)}\}_{k=1}^n$ be the normalized eigenvectors of $\Sigma^{(n)}$, then for any continuous function $f(\cdot)$ defined on $[\delta_1, \delta_2]$, we have

$$\lim_{n \to \infty} \sum_{k=1}^n f(\lambda_k^{(n)}) (\boldsymbol{s}_n^T \boldsymbol{\phi}_k^{(n)})^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(S(\omega)) P(\omega) d\omega.$$

Theorem 2 (Gärtner-Ellis [15]):

Let $\{y_n\} \in \mathbb{R}$ be a sequence of random variables drawn according to the probability law $\{P_n\}$, and define

$$\Lambda^{(n)}(t) = \log \mathbb{E}[e^{ty_n}]. \tag{4}$$

Assumptions: (1) For each $t \in \mathbb{R}$, the logarithmic moment-generating function, defined as the limit $\Lambda(t) = \lim_{n \to \infty} \frac{1}{n} \Lambda^{(n)}(nt)$ exists as an extended real number. (2) The interior of $D_{\Lambda} = \{t \in \mathbb{R} : \Lambda(t) < \infty\}$, denoted by D_{Λ}^{o} , contains the origin. (3) $\Lambda(\cdot)$ is differentiable throughout D_{Λ}^{o} , and (4) $\Lambda(\cdot)$ is steep, i.e., $\lim_{n\to\infty} \Lambda'(t_n) = \infty$ whenever $\{t_n\}$ is a sequence in D_{Λ}^{o} converging to a boundary point of D_{Λ}^{o} . Under the above assumptions, the large deviation principle (LDP) satisfied by the sequence $\{P_n\}$ can be characterized by the Fenchel-Legendre transform of $\Lambda(t)$:

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}} \{ xt - \Lambda(t) \}.$$
(5)

That is, if G^o and \overline{G} are the interior and closure of a set $G \subset \mathbb{R}$, respectively, we say that $\{y_n\}$ satisfies the LDP with rate function $\Lambda^*(x)$ if, for any $G \subset \mathbb{R}$ we have

$$-\inf_{x\in G^{o}}\Lambda^{*}(x) \leq \lim\inf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(y_{n}\in G)$$
$$\leq \lim\sup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(y_{n}\in G)\leq -\inf_{x\in\bar{G}}\Lambda^{*}(x).$$
(6)

²The scaling factor to satisfy the power constraint is assumed to be absorbed by the measurements $\{x_k\}$.

In hypothesis testing G mostly satisfies the Δ -continuous property [15] where ϵ_G is the error exponent:

$$\epsilon_G \triangleq \inf_{x \in G^o} \Lambda^*(x) = \inf_{x \in \bar{G}} \Lambda^*(x). \tag{7}$$

IV. DETECTION ERROR EXPONENTS

Within the Neyman-Pearson framework, the probability of false alarm P_{fa} is defined as the probability of choosing hypothesis \mathcal{H}_1 when \mathcal{H}_0 is correct, and the probability of miss detection P_m is defined as the probability of deciding \mathcal{H}_0 when \mathcal{H}_1 is true. In this work, we are interested in the analysis of large networks, and therefore, we analyze the rate of decaying of both P_{fa} and P_m when the number of sensor *n* approaches infinity, i.e., we compute the corresponding error exponents ϵ_{fa} and ϵ_m defined as,

$$\epsilon_{fa} = -\lim_{n \to \infty} \frac{1}{n} \log P_{fa}^{(n)}, \quad \epsilon_m = -\lim_{n \to \infty} \frac{1}{n} \log P_m^{(n)}.$$

Let $G_0 = \{x \in \mathbb{R} : x \leq \tau\}$ and $G_1 = \{x \in \mathbb{R} : x > \tau\}$ denote the decision regions. The detector decides \mathcal{H}_0 if $y_n \in G_0$ and \mathcal{H}_1 if $y_n \in G_1$. We compute the error exponent ϵ_{fa} by applying Th. 2 with $G = G_1$ and considering the expectation in (4) under \mathcal{H}_0 . Similarly for ϵ_m , we use $G = G_0$ and take the expectation in (4) under \mathcal{H}_1 . Then, for hypothesis \mathcal{H}_i , i = 0, 1, (4) becomes

$$\Lambda_{i}^{(n)}(nt) = \log \mathbb{E}_{Y_{n}|\mathcal{H}_{i}}[e^{nty_{n}}], \qquad i = 0, 1.$$
$$= \log \mathbb{E}_{\boldsymbol{X}_{n}|\mathcal{H}_{i}}\left[\prod_{k=1}^{n} \mathbb{E}_{H_{k}|X_{k}=x_{k}}\left(e^{th_{k}x_{k}}|\mathcal{H}_{i}\right)\right]$$
$$+ \log \mathbb{E}_{W}\left(e^{tw}\right), \qquad (8)$$

where in the second step we have used the fact that the fading coefficients are independent. Considering that both h_k and w are Gaussian random variables, we use their moment-generating function to write

$$\Lambda_i^{(n)}(nt) = \log \mathbb{E}_{\boldsymbol{X}_n \mid \mathcal{H}_i} \left[e^{t\mu_h \boldsymbol{1}_n^T \boldsymbol{x}_n + \frac{\sigma_h^2}{2} t^2 \|\boldsymbol{x}_n\|^2} \right] + \frac{t^2 \sigma_w^2}{2}.$$
(9)

The distribution of X_n is a multivariate Gaussian with covariance matrix $\Sigma^{(n)}$ and mean $\mu_i \mathbf{1}_n$, where $\mu_0 = 0$ under \mathcal{H}_0 and $\mu_1 = \theta_0$ under \mathcal{H}_1 . Completing squares in (9) we obtain

$$\Lambda_i^{(n)}(nt) = \frac{a_i^{(n)}}{2} - \frac{1}{2} \sum_{k=1}^n \log\left(1 - t^2 \sigma_h^2 \lambda_k^{(n)}\right) + \frac{t^2 \sigma_w^2}{2}, \quad (10)$$

where $\lambda_k^{(n)}$ are the eigenvalues of $\boldsymbol{\Sigma}^{(n)}$ and

$$a_i^{(n)} = \mathbf{1}_n^T \left(t\mu_h \mathbf{I}_n + \mu_i \boldsymbol{\Sigma}^{(n)-1} \right) \left(\boldsymbol{\Sigma}^{(n)-1} - t^2 \sigma_h^2 \mathbf{I}_n \right)^{-1} \times \left(t\mu_h \mathbf{I}_n + \mu_i \boldsymbol{\Sigma}^{(n)-1} \right) \mathbf{1}_n - \mu_i^2 \mathbf{1}_n^T \boldsymbol{\Sigma}^{(n)-1} \mathbf{1}_n$$

with I_n being the identity matrix of size $n \times n$. Considering Th. 1 with $P(\omega) = 2\pi\delta(\omega)$, which is the spectral density of the unit-energy constant sequence $1/\sqrt{n}, \ldots, 1/\sqrt{n}$, and using $f(\cdot) = \log(\cdot)$, we have

$$\Lambda_{i}(t) = \lim_{n \to \infty} \frac{1}{n} \Lambda_{i}^{(n)}(nt)$$

= $\frac{1}{2} \frac{t^{2} \mu_{h}^{2} S_{z}(0) + 2t \mu_{h} \mu_{i} + t^{2} \sigma_{h}^{2} \mu_{i}^{2}}{1 - t^{2} \sigma_{h}^{2} S_{z}(0)}$
+ $\frac{1}{4\pi} \int_{-\pi}^{\pi} -\log\left(1 - t^{2} \sigma_{h}^{2} S_{z}(\omega)\right) d\omega,$ (11)

where $S_z(\omega)$ is the power spectral density of the wide-sense stationary process $\{z_k\}$. The effective domain of $\Lambda_i(t)$ is $D_{\Lambda_i} = \{t \in \mathbb{R} : |t| < 1/(\sigma_h \sqrt{\max S_z(\omega)})\}$. Note that in the first term of the expression above the power spectral density appears only trough its DC component. This is because the deterministic signal considered is constant. To calculate the error exponents, the optimization (5) can be done numerically.

In hypothesis testing, we typically have that the threshold $\tau \in (\mathbb{E}(y_n | \mathcal{H}_0), \mathbb{E}(y_n | \mathcal{H}_1))$. This is because for τ laying out of that interval the error exponents cannot be non-zero simultaneously and the practical interest is when both, the false alarm and miss error probabilities decay exponentially to zero. Therefore, and because of the convexity of $\Lambda_i(t)$, the last optimization given by (6) is achieved for $x^* = \tau$ and the error exponents are given by $\Lambda_i^*(\tau)$ [16].

Let K be the strength of the LOS component divided by scattered power, i.e., $K = \mu_h^2/\sigma_h^2$. We normalize the second order moment of the fading in such a way that $\mu_h^2 + \sigma_h^2 = 1$. When K approaches zero, the mean of the channel goes to zero and the LOS component vanishes. On the other hand, if K approaches infinity, σ_h^2 goes to zero and the channel becomes deterministic. It is worth to note that the case of a MAC channel without fading can be obtained by taking the limit of (11) when $K \to \infty$ (or $\sigma_h^2 \to 0$). The Fengel-Legendre transform can be computed analytically and we recover the result in [1] as a particular case:

$$\epsilon_{fa} = \frac{\theta_0^2}{2S_z(0)}\tau^2, \quad \epsilon_m = \frac{\theta_0^2}{2S_z(0)}(1-\tau)^2 \quad 0 \le \tau \le 1,$$
 (12)

where the threshold was normalized to one. Both error exponents depend on $S_z(0)$, which is proportional to the variance of the process $\{z_k\}$, σ_z^2 . Therefore, the error exponents depend proportionally on the signal-to-noise ratio defined as $\text{SNR} = \theta_0^2/\sigma_z^2$. The trade-off between ϵ_{fa} and ϵ_m is evident in (12).

V. NUMERICAL RESULTS

In this section, we evaluate numerically the error exponents for an autoregressive process of order 1 with power spectral density given by $S_z(\omega) = \frac{\sigma_z^2(1-\rho^2)}{1+\rho^2-2\rho\cos(\omega)}$ where $\rho \in (-1,1)$ determines the correlation of the process.

In Fig. 1, the receiver operating characteristic (ROC) for the error exponents is plotted for several SNR's for the indicated K and ρ . We observe the typical trade-off between both error exponents when the threshold of the test is varied. Both ϵ_m and ϵ_{fa} exhibit a saturation effect with the SNR due to the fading, although the saturation of the false alarm exponent is at a higher SNR. That is, for a given K, there exists an SNR for each error exponent from which they do not increase significantly. Note that in the non-fading case, the saturation effect disappears because of $K \to \infty$.

In Fig. 2, both error exponents are plotted vs. the correlation ρ . There exists an optimal correlation in which ϵ_{fa} or ϵ_m are maximum but generally not for the same ρ^* . Note that when $|\rho|$ approaches 1, the error exponents go to zero. In the case of $|\rho| = 1$, the covariance matrix of the process has rank one and therefore, the process z_n is determined by a unique random variable. Then, the received signal y_n can be expressed asymptotically as the product of two independent Gaussian



Figure 1. ROC for error exponents of the detector with $\rho = 0.6$, K = 1 and the SNR (in dB) as a parameter.

random variables, the one corresponding to the process and the other corresponding to the channel. By computing its distribution, it is easy to show that both error exponents are null.



Figure 2. False alarm (cont.) and miss detection (dashed) error exponents as a function of the correlation coefficient with K = 1, and SNR 0 and 15dB and normalized threshold $\tau = 0.7$.

In Fig. 3, both error exponents are plotted against K, the LOS strength of the fading channel for the parameters indicated in the figure. As it is shown, the fading is harmful and make the error exponents decrease. This is evident for low values of K, where the fading variance is high with respect to its mean. However, the error exponents saturates for relatively low values of K and the fading channel behaves as a deterministic channel: $K_{sat} \approx 1$ for SNR = 0dB and $K_{sat} \approx 10$ for SNR = 15dB. In this scenario, the performance coincides with that predicted by (12).

VI. CONCLUSIONS

In this paper, we analyze a distributed detection problem in the context of a wireless sensor network with a FC. We derived and evaluated the error exponents for the Neyman-Pearson framework using LDT for a MAC Rician fading channel. We showed how the error exponents behave with the process correlation and that the error exponents exhibit a saturation effect because of the fading characteristics. The general conclusion is that, although fading degrades the system performance, the analyzed scheme still works under non-zeromean fading channels and converges to the optimal centralized scheme even when the fading LOS component is not so strong.



Figure 3. False alarm (cont.) and miss detection (dashed) error exponents as a function of the fading LOS strength for $\rho = 0.6$, $\tau = 0.7$ (normalized threshold) and SNR 0 and 15dB.

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