Source Number Detection with Nested Arrays and ULAs Using Jackknifing

Keyong Han and *Arye Nehorai*

Preston M. Green Department of Electrical and Systems Engineering
Washington University in St. Louis
One Brookings Drive, St. Louis, MO 63130, USA
Email: {kyhannah,nehorai}@ese.wustl.edu
Phone: 314-935-7520 Fax: 314-935-7500

***Abstract*—We consider the problem of source number detection based on uniform linear arrays (ULAs) and the recently proposed nested arrays. A ULA with N sensors can detect at most $N-1$ sources, whereas a nested array provides $O(N^2)$ degrees of freedom with $O(N)$ sensors, enabling us to detect K sources with $N<K$ sensors. In order to make full use of the available limited valuable data, we propose a novel strategy, which is inspired by the jackknifing resampling method. Exploiting numerous iterations of subsets of the whole data set, this strategy helps the existing detection methods achieve great improvements. Numerical simulations demonstrate the advantage of our strategy, both for ULAs and nested arrays.**

***Index Terms*—Source number detection, nested array, jackknifing, uniform linear array**

I. Introduction

Source number detection is a prerequisite for direction of arrival (DOA) estimation. The use of a ULA for source number detection has received a considerable amount of attention in the last three decades [1]-[8]. Various methods have been proposed according to different mathematical criteria. The most commonly used techniques are based on information theoretic criteria, such as the Akaike information criterion (AIC) [3], the Kullback-Leibler information criterion (KIC) [4], and Rissanen's minimum description length (MDL) [5] principle. These methods conduct detection by combining eigenvalue decomposition, the maximum likelihood function, and some penalty functions. Another eigenvalue-based method, called second order statistic of eigenvalues (SORTE) [6], is based on a gap measure of the eigenvalues. A predicted eigenthreshod (ET) approach was proposed by Chen [7], which detects the number of sources by setting an upper bound on the eigenvalues and then implementing a hypothesis testing procedure. All the aforementioned methods are based on eigenvalues of the sample covariance matrix. Eigenvectors can also be used for the determination of sources. Jiang and Ingram [8] proposed an eigenvector-based method by exploiting the property of the variance of the rotational submatrix (VTRS).

All the existing detection methods exploit the available data by calculating the whole sample covariance matrix. However, this fails to make full use of the available limited information. Jackknifing [9] is a resampling strategy used to estimate sample statistics by using subsets of the available data. Jackknifing [10] helps fully exploit the received data to improve the detection performance.

A ULA with N sensors can detect at most $N-1$ sources. A systematic approach to achieve $O(N^2)$ degrees of freedom using $O(N)$ sensors based on a nested array was recently proposed in [11], where DOA estimation and beamforming were studied. This nested array can detect more sources than the number of sensors [12].

In this paper, we propose a novel strategy, jackknifing, to detect the number of sources. Both the ULA and the nested-array are considered. Simulations are provided to demonstrate the advantage of our strategy.

II. Signal Model

We assume there is a nonuniform linear nested array with N sensors, including two concatenated uniform linear arrays (ULA). Suppose the inner ULA has N_1 sensors with spacing d_I and the outer ULA has N_2 sensors with spacing $d_O = (N_1+1)d_I$. We assume K narrowband sources are in the surveillance region, impinging on this linear array from directions $\{\theta_k, k = 1,\dots,K\}$. We can obtain the received signal as

$$\boldsymbol{y}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{e}(t), \tag{1}$$

where $\boldsymbol{y}(t) = [y_1(t), y_2(t), \dots, y_N(t)]^T$ is the N received signal at time t. Let $\boldsymbol{a}(\theta_k)$ be the $N \times 1$ steering vector with the ith element $e^{j(2\pi/\lambda)d_i \sin\theta_k}$. d_i is the position of ith sensor, which is an integer multiple of the basic spacing d_I or d_O. λ denotes the carrier wavelength. Then the manifold matrix can be expressed as $\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_2), \dots, \boldsymbol{a}(\theta_K)]$. $\boldsymbol{x}(t) = [x_1(t), x_2(t), \dots, x_K(t)]^T$ is the source vector. We suppose the source signals follow Gaussian distributions, $x_k \sim \mathcal{N}(0, \sigma_k^2)$, and they are all independent of each other. The noise signal $\boldsymbol{e}(t) = [e_1(t), e_2(t), \dots, e_N(t)]^T$ is assumed to be white Gaussian, and uncorrelated with the sources.

Suppose we have T measurements. Stacking all the measurements together, we rewrite (1) as

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{E}, \tag{2}$$

This work was supported by the AFOSR Grant FA9550-11-1-0210 and ONR Grant N000141310050.

where

- $\boldsymbol{Y} = [\boldsymbol{y}(1), \boldsymbol{y}(2), \ldots, \boldsymbol{y}(T)]$, an $N \times T$ matrix,
- $\boldsymbol{X} = [\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(T)]$, an $K \times T$ matrix,
- $\boldsymbol{E} = [\boldsymbol{e}(1), \boldsymbol{e}(t), \ldots, \boldsymbol{e}(T)]$, an $N \times T$ matrix.

Based on our assumption, the source autocorrelation matrix $\boldsymbol{R_x}$ is diagonal: $\boldsymbol{R_x} = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2)$. Then the autocorrelation matrix of the received signal is

$$\boldsymbol{R_y} = \boldsymbol{A}\boldsymbol{R_x}\boldsymbol{A}^H + \sigma_e^2\boldsymbol{I},$$

where σ_e^2 is the noise power, and $\boldsymbol{I}$ is the identity matrix. Vectorizing $\boldsymbol{R_y}$, we get:

$$\boldsymbol{v} = (\boldsymbol{A}^* \otimes \boldsymbol{A})\boldsymbol{p} + \sigma_e^2\boldsymbol{1}_e, \tag{3}$$

where $\boldsymbol{p} = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2]^T$, and $\boldsymbol{1}_e = [\boldsymbol{e}_1^T, \boldsymbol{e}_2^T, \ldots, \boldsymbol{e}_N^T]^T$, with $\boldsymbol{e}_i$ being a vector of all zeros except a 1 at the ith position. We can view vector $\boldsymbol{v}$ in (3) as some new longer received signals with the new manifold matrix $\boldsymbol{A}^* \otimes \boldsymbol{A}$, and the new source signals $\boldsymbol{p}$. $*$ denotes conjugation without transpose, and $\otimes$ denotes the Khatri-Rao product.

III. Source detection

We will use the nested array mentioned above to conduct detection of more sources than the number of sensors. First, we will briefly introduce spatial smoothing [11], which is used to exploit the increased degrees of freedom. Then, we will list four source detection methods, and propose a new strategy, which applies jackknifing to improve the detection performance.

A. Spatial Smoothing

In order to exploit the increased degrees of freedom provided by the co-array, we need to apply spatial smoothing. We remove the repeated rows from $\boldsymbol{A}^* \otimes \boldsymbol{A}$ and also sorte them so that the ith row corresponds to the sensor location $(-N^2/4 - N/2 + i)d_I$ in the difference co-array of the 2-level nested array, giving a new vector: $\bar{\boldsymbol{v}} = \bar{\boldsymbol{A}}\boldsymbol{p} + \sigma_e^2\bar{\boldsymbol{e}}$, where $\bar{\boldsymbol{e}} \in \mathbb{R}^{((N^2-2)/2+N)\times 1}$ is a vector of all zeros except a 1 at the center position.

The difference co-array of this 2-level nested array has sensors located at

$$(-N^2/4 - N/2 + 1)d_I, \ldots, -d_I, 0, d_I, \ldots, (N^2/4 + N/2 - 1)d_I.$$

We now divide these $N^2/2 + N - 1$ sensors into $N^2/4 + N/2$ overlapping subarrays, each with $N^2/4 + N/2$ elements, where the ith subarray has sensors located at $\{(-i + 1 + n)d_I,\ n = 0, 1, \ldots, \frac{N^2}{4} + \frac{N}{2} - 1\}$. The ith subarray corresponds to the $(N^2/4 + N/2 - i + 1)$th to $(N^2 + N - i)$th rows of $\bar{\boldsymbol{v}}$, denoted as $\bar{\boldsymbol{v}}_i = \bar{\boldsymbol{A}}_i\boldsymbol{p} + \sigma_e^2\boldsymbol{e}_i$. We can check that $\bar{\boldsymbol{v}}_i = \bar{\boldsymbol{A}}_1\boldsymbol{\Phi}^{i-1}\boldsymbol{p} + \sigma_e^2\boldsymbol{e}_i$, where $\boldsymbol{\Phi} = \mathrm{diag}(e^{-j(2\pi/\lambda)d\sin\theta_1}, e^{-j(2\pi/\lambda)d\sin\theta_2}, \ldots, e^{-j(2\pi/\lambda)d\sin\theta_D})$. Viewing $\bar{\boldsymbol{v}}_i$ as a newly received vector, we can get the equivalent covariance matrix $\boldsymbol{R}_i = \bar{\boldsymbol{v}}_i\bar{\boldsymbol{v}}_i^H$. Taking the average of $\boldsymbol{R}_i$, we get

$$\boldsymbol{R}_{\mathrm{ave}} = \frac{1}{(\frac{N^2}{4} + \frac{N}{2})} \sum_{i=1}^{N^2/4+N/2} \boldsymbol{R}_i. \tag{4}$$

The spatially smoothed matrix $\boldsymbol{R}_{\mathrm{ave}}$ enables us to perform detection of $O(N^2)$ sources with N sensors.

B. Source Detection Using Jackknifing

As mentioned in the introduction, the sample covariance matrix $\boldsymbol{R_y}$ is a key element for source detection. Considering a uniform linear array with N sensors, we do eigenvalue decomposition: $\mathrm{EVD}(\boldsymbol{R_y}) = \boldsymbol{U}\Lambda\boldsymbol{U}^T$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ are the eigenvalues and $\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N]$ is the corresponding eigenvector matrix. Suppose the eigenvalues are sorted decreasingly:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_K > \lambda_{K+1} = \ldots = \lambda_N.$$

We now briefly introduce four detection methods.

- SORTE
  A gap measure is defined:
  $$\mathrm{SORTE}(k) = \begin{cases} \frac{\mathrm{var}(\{\nabla\lambda_i\}_{i=k+1}^{N-1})}{\mathrm{var}(\{\nabla\lambda_i\}_{i=k}^{N-1})}, & \mathrm{var}(\{\nabla\lambda_i\}_{i=k}^{N-1}) \neq 0 \\ +\infty & \mathrm{var}(\{\nabla\lambda_i\}_{i=k}^{N-1}) = 0 \end{cases}$$
  where $k = 1, \ldots, N-2$, $\nabla\lambda_i = \lambda_i - \lambda_{i+1}$ and
  $$\mathrm{var}(\{\nabla\lambda_i\}_{i=k}^{N-1}) = \frac{1}{N-k}\sum_{i=k}^{N-1}(\nabla\lambda_i - \frac{1}{N-k}\sum_{j=k}^{N-1}\nabla\lambda_j)^2.$$
  Then the source number is $K = \arg\min_k \mathrm{SORTE}(k)$.
- VTRS
  Suppose E_s is the combined signal eigenvectors of $\boldsymbol{R_y}$, and E_x and E_y are the first $N-1$ rows and last $N-1$ rows of E_s respectively. Solving $\mathrm{E}_y = \mathrm{E}_x\Phi$ based on the least square criterion, we get matrix Φ. Define $\Delta_k = \{\Phi(i,j)\}_{(N-k-1)\times k},\ i = k+1, \ldots, N-1, j = 1, \ldots, k$. Then the source number is
  $$K = \arg\min_k \frac{\|\Delta_k\|^2}{(N-k-1)K},\ k = 1, \ldots, N-2,$$
  where $\|\cdot\|$ is the Frobenius norm.
- ET
  Define the eigen-threshold
  $$\bar{\lambda}_{N-m} = [(m+1)\frac{1 + t(T(m+1))^{-1/2}}{1 - t(Tm)^{-1/2}} - m]l_{N-m+1}, \tag{5}$$
  where t is a pre-set parameter, and
  $$l_i = \frac{1}{N-i+1}\sum_{j=i}^{N}\lambda_j,\ i = k+1, \ldots, N.$$
  Based on this, we keep testing the binary hypothesis: H_0 : $K < N-m$ and H_1 : $K = N-m$. Accept H_1 or H_0 according to $\lambda_{N-m} \underset{>H_0}{\overset{\leq H_1}{}} \bar{\lambda}_{N-m}$. If H_0 is accepted, then we set $m = m+1$, and continue. Otherwise, if H_1 is accepted, stop testing, and assign $K = N-m$.
- AIC
  Define
  $$L(k) = \frac{T}{2}\log\Big(\frac{\prod_{i=k+1}^{N}\lambda_i^{1/(N-k)}}{\frac{1}{N-k}\sum_{i=K+1}^{N}\lambda_i}\Big)^{N-k},$$

and $P(k) = 1 + Nk - 1/2k(k-1)$. Then the source number is determined as

$$K = \arg\ \min_k\ -2L(k) + 2P(k).$$

All the existing methods are based on the eigenvalues or eigenvectors of $\boldsymbol{R_y}$. However, the received data can tell us more.

Researchers have been using all the measurements as a whole to get the sample covariance, then studying further based on this covariance matrix. What we will try to do here is make full use of the received data, achieving more accurate detection. Jackknifing is an effective strategy used in statistical area to estimate sample statistics. The idea is to use subsets of available data to improve performance. We propose a new detection strategy based on the idea of jackknifing. Our basic belief is that a large proportion of the available data approximately contains the same amount of information as the whole available data set does.

Suppose we have T measurements in total:

$$\boldsymbol{Y} = [\boldsymbol{y}(1), \boldsymbol{y}(2), \ldots, \boldsymbol{y}(T)].$$

First, we take a subset $\boldsymbol{Y}_J$ of size T_J from the T measurements matrix $\boldsymbol{Y}$:

$$\boldsymbol{Y}_J = [\hat{\boldsymbol{y}}(1), \hat{\boldsymbol{y}}(2), \ldots, \hat{\boldsymbol{y}}(T_J)],$$

where $\boldsymbol{Y}_J \subset \boldsymbol{Y}$, $\hat{\boldsymbol{y}}(t) \in \boldsymbol{Y}$, and $T_J = rT$, with r expressed as a percentage and satisfying $0.5 < r < 1$. The constraints for r help to guarantee our basic belief that the subsets contain enough information. Specifically, we randomly pick T_J elements from $\boldsymbol{Y}$, without replacement, to form $\boldsymbol{Y}_J$. The sample covariance based on $\boldsymbol{Y}_J$ is

$$\begin{aligned}\boldsymbol{R}_{\boldsymbol{Y}_J} &= \frac{1}{T_J}\boldsymbol{Y}_J\boldsymbol{Y}_J^H \\ &= \frac{1}{T_J}\sum_{t=1}^{T_J}\hat{\boldsymbol{y}}(t)\hat{\boldsymbol{y}}(t)^H.\end{aligned}$$

Then we do eigenvalue decomposition for $\boldsymbol{R}_{\boldsymbol{Y}_J}$:

$$\mathrm{EVD}(\boldsymbol{R}_{\boldsymbol{Y}_J}) = \boldsymbol{U}_J\Lambda_J\boldsymbol{U}_J^T,$$

where $\Lambda_J = \mathrm{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_{T_J})$. Using Λ_J and $\boldsymbol{U}_J$, we conduct source detection using the existing methods. Suppose we obtain the source number $\hat{K}$. We continue the above two procedures for Z iterations, obtaining Z estimated source numbers, $\hat{K}_z,\ z = 1, \ldots, Z$. Before making the decision of the final source number, we need one more step, counting the occurrence of each estimated number, denoted as $Z_1, Z_2, \ldots, Z_{N-1}$, with summation Z. The final source number is chosen as the one that occurs most frequently:

$$\hat{K} = \arg\ \max_k\ Z_k. \tag{6}$$

The algorithm is shown in Table I.

TABLE I: Algorithm for Source Detection Using Jackknifing

begin
  $i = 0$; % Iteration counter
  Obtain r, T_J, $\boldsymbol{Y}$
  do
    Randomly pick T_J^i measurements from $\boldsymbol{Y}$, get $\boldsymbol{Y}_{T_J}^i$;
    Obtain the covariance $\boldsymbol{R}_{\boldsymbol{Y}_J}^i$ of $\boldsymbol{Y}_{T_J}^i$;
    Conduct source detection using $\boldsymbol{R}_{\boldsymbol{Y}_J}^i$;
    Obtain the estimated number K^i;
    $i := i + 1$;
  until $i = Z$ % Z is a pre-set threshold;
  Count the number of occurrence of each K^i: Z_{K^i};
  Decide the source number according to (6).
end

Remarks:

- Theoretically, when the detection accuracy is higher than 50%, the detection performance with jackknifing will definitely be improved. Please see the detailed proof in our paper [10]. With over 50% accuracy, the correct number detected should occur more frequently than other numbers do. This assumption is based on the condition that the jackknifing sample subset contains enough information to guarantee over 50% accuracy. Otherwise, the jackknifing will lose its power.
- For methods that are sensitive to sample number, we need to increase the sample number to guarantee the efficiency of jackknifing. For example, one method performs well with the whole T samples. However, when applying jackknifing, we just use rT samples, in which case this method may achieve accuracy lower than 50%. Consequently, jackknifing will fail. Alternatively, we can adjust the value of r to guarantee the accuracy.
- When there is a low signal-to-noise ratio (SNR), namely a high noise level, the detection methods may fail to detect the source number correctly, with accuracy lower than 50%. This will cause jackknifing to perform badly, as discussed in the first remark.

IV. Numerical Examples

In this section, we use numerical examples to show the superiority of our proposed strategy for source detection for both a ULA and a nested array.

A. Uniform Linear Array

We consider a uniform linear array with $N = 8$ sensors with equal spacing $d = \lambda/2$. Suppose we have $K = 3$ sources in the scene, with the same powers $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 9$ and impinging from directions $\boldsymbol{\theta} = [-60^0, 0^0, 30^0]$. We choose a jackknifing iteration number $Z = 20$, the percentage of $r = 0.85$, and the Monte Carlo simulation number $T = 1500$. Fig. 1 shows the results of the aforementioned four different methods: SORTE, VTRS, ET, and AIC, with and without jackknifing. It describes the detection accuracy with respect to the SNR. We take the SNR as $10\log_{10}\frac{\mathrm{E}[x^2]}{\mathrm{E}[e^2]}$, and the detection accuracy as $F_{\hat{K}}/F$, where F is the trial number, and $F_{\hat{K}}$ is the number of times that $\hat{K}$ is detected. In this example, we use $F = 1000$ trials.

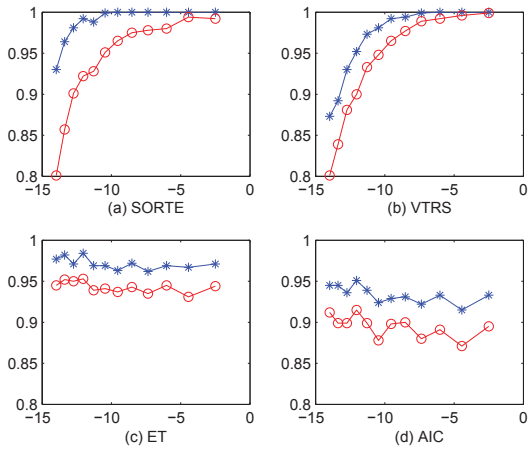

Fig. 1: Performance comparison of four methods with a ULA: the blue-star line is the performance with jackknifing, and the red-circle line without jackknifing. The vertical axis represents the detection accuracy, while the horizontal axis represents SNR.

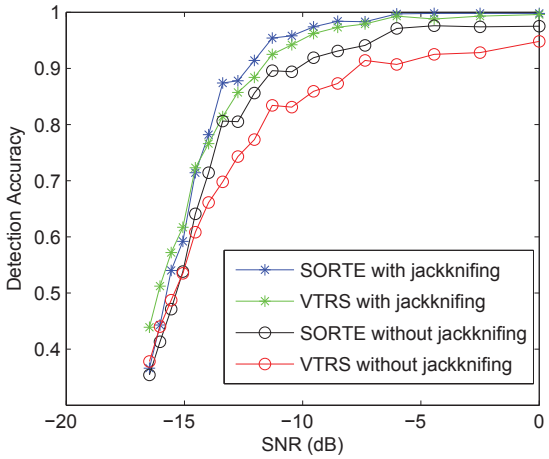

Fig. 2: Performance comparison of SORTE and VTRS with nested arrays using 2000 samples.

We can see that all four methods achieve different levels of improvement by applying jackknifing. SORTE improves the most, and performs even better when the SNR is low. Note that the detection accuracy is above 0.5 without jackknifing, which guarantees the improvement of jackknifing. For ET, the performance is highly related to the appropriate choice of parameter t in (5). In our example, $t = 1.2$.

We also calculated the detection accuracy with respect to various numbers of samples. The results are similar.

B. Nested Array

We consider a 2-level nested array with $N = 6$ sensors, both inner and outer ULAs having 3 sensors, with spacing $d_I = \lambda/2$ and $d_O = 4d_I$. Suppose there are 8 sources, with equal power and impinging from directions $\boldsymbol{\theta} = [-60^0, -45^0, -15^0, 0^0, 15^0, 30^0, 45^0, 60^0]$. It is impossible for us to use a 6-ULAs to detect 8 sources. However the spatial matrix $\boldsymbol{R}_{\text{ave}}$ in (4) helps a nested array obtain this goal. We choose jackknifing iteration number $Z = 20$, and the percentage $r = 0.85$. We conduct $T = 2000$ Monte Carlo simulations, and use $F = 100$ trials. From Fig. 1, we can see SORTE and VTRS perform better, thus we consider only these two methods for the nested array.

Fig. 2 shows the performance of SORTE and VTRS, with and without jackknifing. We can see that at high SNR both methods can detect the source number correctly with high probability. Moreover, with jackknifing, both methods' detection accuracy increases at high SNRs, where the original detection accuracy is higher than 0.5. On the other hand, jackknifing's performance degrades at low SNR, where the accuracy is lower than half. This is in accordance with our previous analysis. Additionally, SORTE slightly outperforms VTRS.

V. Conclusion

In this paper, we proposed a novel strategy for source detection by applying jackknifing. This strategy helps the existing detection methods achieve great improvements by making full use of the limited available data. Numerical examples demonstrate the effectiveness of our strategy. For future work, we will investigate the performance effect of the percentage we choose when doing jackknifing. We will also consider applying this strategy to a co-prime array.

References

[1] A. Di and L. Tian, "Matrix decomposition and multiple source location," in *Acoustics, Speech, and Signal Processing, IEEE International Conference on ICASSP*, Mar. 1984, pp. 722–725.

[2] A. P. Liavas and P. A. Regalia, "On the behavior of information theoretic criteria for model order selection," *IEEE Trans. Signal Process.*, vol. 49, pp. 1689–1695, Aug 2001.

[3] H. Akaike, "A new look at the statistical model identification," *IEEE Trans. Autom. Control*, vol. 19, pp. 716–723, Dec. 1974.

[4] J. E. Cavanaugh, "A large-sample model selection criterion based on Kullback's symmetric divergence," *Stat. Probab. Lett.*, vol. 44, pp. 333–344, 1999.

[5] M. Wax and I. Ziskind, "Detection of the number of coherent signals by the MDL principle," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 37, pp. 1190–1196, Aug 1989.

[6] Z. He, A. Cichocke, S. Xie, and K. Choi, "Detecting the number of clusters in n-way probabilistic clustering," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 32, pp. 2006–2021, Nov 2010.

[7] W. Chen, K. M. Wong, and J. P. Reilly, "Detection of the number of signals: a predicted eigen-threshold approach," *IEEE Trans. Signal Process.*, vol. 39, pp. 1088–1098, May 1991.

[8] J. S. Jiang and M. A. Ingram, "Robust detection of number of sources using the transformed rotational matrix," in *Wireless Communications and Networking Conference*, Mar. 2004, pp. 501–506.

[9] J. Shao and D. S. Tu, *The Jackknifing and Bootstrap*. New York: Springer, 1995.

[10] K. Han and A. Nehorai, "Improved source number detection and direction estimation with nested arrays and ULAs using jackknifing," *IEEE Trans. Signal Process.*, to be published.

[11] P. Pal and P. P. Vaidyanathan, "Nested array: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, pp. 4167–4181, Aug. 2010.

[12] K. Han and A. Nehorai, "Wideband Gaussian source processing using a linear nested array," *IEEE Signal Process. Lett.*, to be published.