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Wideband Direction of Arrival Estimation Using Nested Arrays

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***Abstract*—Based on a newly proposed nested array, we consider the problem of direction of arrival estimation for wideband sources. This array provides $O(N^2)$ degrees of freedom with $O(N)$ sensors, enabling us to estimate K sources with $N<K$ sensors. To employ the nested array for wideband source estimation, we propose a novel strategy to apply nested-array processing to each frequency component, and combine all the spectral information of various frequencies to conduct estimation. Similar to the narrowband case, the nested array achieves great performance for wideband scenarios. Numerical simulations demonstrate the advantage of our strategy.**

***Index Terms*—Direction of arrival estimation, nested array, wideband source**

I. Introduction

Direction of arrival (DOA) estimation is a major application of the antenna array. Theories are well established for narrowband sources, and a large body of literature exists [1]. Owing to the narrowband property, the array model can be greatly simplified [2]. Subspace based methods and the maximum likelihood approach are two main topics. For wideband sources, however, the literature is somewhat scanty. Wax *et al.* is among the earlier researchers in this field [3], decomposing the incoherent wideband signal into many narrowband signals using discrete Fourier transform (DFT) along the temporal domain. Wang and Kaveh [4] considered the case of coherent wideband sources.

However, the DOA estimation is mostly confined to the case of ULAs [5]. A ULA with N sensors can resolve at most $N-1$ sources using conventional subspace-based methods such as multiple signal classification (MUSIC) [6]. A systematic approach to achieve $O(N^2)$ degrees of freedom (DOF) using $O(N)$ sensors based on a nested array was recently proposed in [7], where DOA estimation and beamforming were studied. The nested arrays are obtained by combining two or more ULAs with increasing spacing. Owing to the property of nonuniformity, the resulting difference co-array has significantly more DOF than the original sparse array, which makes it possible for the nested array to detect more sources than the number of sensors [8], [9].

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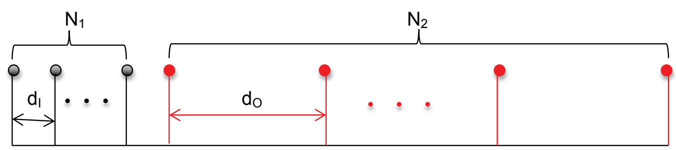

Fig. 1: A 2-level nested array with N_1 sensors in the inner ULA, and N_2 sensors in the outer ULA, with intersensor spacings d_{I} and d_{O} respectively.

In this paper, we consider the incoherent case. We propose a novel strategy to employ nested array for wideband source estimation [10]. Similar to the technique in [3], we decompose the wideband signal into different narrowband components and apply nested array to each of those frequencies. A combined MUSIC spectrum is proposed to exploit all the spectral information from different frequency analysis. Simulations are provided to demonstrate the advantage of our strategy. More precisely, a 2-level nested array is a linear array with sensor locations given by the union of the sets $S_{\mathrm{I}} = \{md_{\mathrm{I}},\ m = 1,\ldots,N_1\}$ and $S_{\mathrm{O}} = \{n(N_1+1)d_{\mathrm{I}},\ n = 1,\ldots,N_2\}$, as shown in Fig. 1.

II. Signal Model

We assume there is a linear nested array with N sensors, including two concatenated uniform linear arrays (ULA). Suppose the inner ULA has N_1 sensors with spacing d_{I} and the outer ULA has N_2 sensors with spacing $d_{\mathrm{O}} = (N_1+1)d_{\mathrm{I}}$.

We assume K wideband sources are in the surveillance region, impinging on this linear array from directions $\{\theta_k, k = 1,\ldots,K\}$. Assume that the incident wideband signals have a common bandwidth B with center frequency f_c. Let $s_k(t)$ denote the kth baseband signal. Then the observed bandpass signal $\bar{x}_k(t)$ at a reference point can be written as

$$\bar{x}_k(t) = s_k(t)e^{j2\pi f_c t}. \tag{1}$$

If we observe the signal over the time interval $[t_1, t_2]$, then the baseband signal can be written as [11]

$$s_k(t) = \sum_{i=1}^{I} S_k(f_i)e^{j2\pi f_i t},\ t_1 \leq t \leq t_2, \tag{2}$$

where $S_k(f_i)$ are the Fourier coefficients

$$S_k(f_i) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s_k(t) e^{-j2\pi f_i t} dt, \tag{3}$$

with $f_i = f_l + (i-1)B/(I-1),\ i = 1, \ldots, I$. f_l denotes the lowest frequency included in the bandwidth B, and I is the number of frequency components. We choose f_l and I so that the frequencies are symmetric about 0 Hz. By considering the propagation delay $\tau_{k,n}$ of the kth signal at the nth sensor, the modulated bandpass signal at the reference point can be presented as

$$\bar{x}_k(t + \tau_{k,n}) = \sum_{i=1}^{I} S_k(f_i) e^{j2\pi(f_c + f_i)(t + \tau_{k,n})}, \tag{4}$$

where $\tau_{k,n} = n d_I \sin(\theta_k)/c$, $k = 1, \ldots, K$, and $n = 1, \ldots, N$, with c being the propagation speed.

The demodulated signal can be expressed as

$$x_k(t, \tau_{k,n}) = \bar{x}_k(t + \tau_{k,n}) e^{-j2\pi f_c t} \tag{5}$$

Stacking $\{x_k(t, \tau_{k,n})\}_{n=1}^{N}$ according to sensor number, we get the $N \times 1$ vector $\boldsymbol{x}_k(t)$. Let $\boldsymbol{a}(\theta_k, f_c + f_i)$ denote the $N \times 1$ steering vector of the kth source and the ith frequency component:

$$\boldsymbol{a}(\theta_k, f_i) = [e^{j2\pi(f_c+f_i)\tau_{k,1}}, \ldots, e^{j2\pi(f_c+f_i)\tau_{k,N}}]^T. \tag{6}$$

Then the received data vector has the form

$$\boldsymbol{x}(t) = \sum_{k=1}^{K} \boldsymbol{x}_k(t) = \sum_{i=1}^{I} [\boldsymbol{A}(\boldsymbol{\theta}, f_i)\boldsymbol{S}(f_i) + \boldsymbol{E}(f_i)] e^{j2\pi f_i t}, \tag{7}$$

where $\boldsymbol{A}(\boldsymbol{\theta}, f_i) = [\boldsymbol{a}(\theta_1, f_i), \ldots, \boldsymbol{a}(\theta_K, f_i)]$, $\boldsymbol{S}(f_i) = [S_1(f_i), \ldots, S_K(f_i)]^T$ is the $K \times 1$ signal vector, and $\boldsymbol{E}(f_i) = [E_1(f_i), \ldots, E_K(f_i)]^T$ is the $N \times 1$ noise Fourier coefficient vector. Define

$$\boldsymbol{y}(i) \triangleq \boldsymbol{A}(\boldsymbol{\theta}, f_i)\boldsymbol{S}(f_i) + \boldsymbol{E}(f_i),\ i = 1, \ldots, I. \tag{8}$$

Then $\{\boldsymbol{y}(i)\}$ are by definition the $N \times 1$ Fourier coefficient vectors of $\boldsymbol{x}(t)$.

We assume the source signals follow Gaussian distributions, $S_k(f_i) \sim \mathcal{N}(0, \sigma_{k,i}^2)$, and that they are all independent of each other. The noise $\boldsymbol{E}(f_i)$ is assumed to be white Gaussian and uncorrelated with the sources. Based on our assumption, the source autocorrelation matrix $\boldsymbol{R}_{\boldsymbol{s}_i}$ is diagonal: $\boldsymbol{R}_{\boldsymbol{s}_i} = \mathrm{diag}(\sigma_{1,i}^2, \sigma_{2,i}^2, \ldots, \sigma_{K,i}^2)$. We use $\boldsymbol{A}_i$ to represent $\boldsymbol{A}(\boldsymbol{\theta}, f_i)$ for brevity. Then the autocorrelation matrix of $\{\boldsymbol{y}(i)\}$ is

$$\boldsymbol{R}_{\boldsymbol{y}_i} = \boldsymbol{A}_i \boldsymbol{R}_{\boldsymbol{s}_i} \boldsymbol{A}_i^H + \sigma_E^2 \boldsymbol{I},$$

where σ_E^2 is the noise power, and $\boldsymbol{I}$ is the identity matrix. Vectorizing $\boldsymbol{R}_{\boldsymbol{y}_i}$, we get

$$\boldsymbol{v}_i = (\boldsymbol{A}_i^* \otimes \boldsymbol{A}_i)\boldsymbol{p}_i + \sigma_E^2 \boldsymbol{1}_e, \tag{9}$$

where $\boldsymbol{p}_i = [\sigma_{1,i}^2, \ldots, \sigma_{K,i}^2]^T$, and $\boldsymbol{1}_e = [\boldsymbol{e}_1^T, \boldsymbol{e}_2^T, \ldots, \boldsymbol{e}_N^T]^T$, with $\boldsymbol{e}_i$ being a vector of all zeros except a 1 at the ith position. We can view vector $\boldsymbol{v}_i$ in (9) as some new longer received signals with the new manifold matrix $\boldsymbol{A}_i^* \otimes \boldsymbol{A}_i$, and the new source signals $\boldsymbol{p}_i$. The symbol $*$ denotes conjugation without transpose, and $\otimes$ denotes the Khatri-Rao product.

III. Direction of Arrival Estimation

We will use the nested array mentioned above to conduct direction of arrival estimation, where the source number is greater than the sensor number. First, we will briefly introduce spatial smoothing [7], which is used to exploit the increased degrees of freedom. Then, we will propose a novel strategy for wideband source estimation using MUSIC.

A. Spatial Smoothing

To exploit the increased degrees of freedom provided by the co-array, we need to apply spatial smoothing. We remove the repeated rows from $\boldsymbol{A}_i^* \otimes \boldsymbol{A}_i$ and also sort them so that the jth row corresponds to the sensor location $(-N^2/4 - N/2 + j)d_I$ in the difference co-array of the 2-level nested array, giving a new vector:

$$\bar{\boldsymbol{v}}_i = \bar{\boldsymbol{A}}_i \boldsymbol{p}_i + \sigma_E^2 \bar{\boldsymbol{e}},$$

where $\bar{\boldsymbol{e}} \in \mathbb{R}^{((N^2-2)/2+N)\times 1}$ is a vector of all zeros except a 1 at the center position.

The difference co-array of this 2-level nested array has sensors located at

$$(-N^2/4 - N/2 + 1)d_I, \ldots, -d_I, 0, d_I, \ldots, (N^2/4 + N/2 - 1)d_I.$$

We now divide these $N^2/2 + N - 1$ sensors into $N^2/4 + N/2$ overlapping subarrays, each with $N^2/4 + N/2$ elements, where the lth subarray has sensors located at $\{(-l + 1 + n)d_I,\ n = 0, 1, \ldots, \frac{N^2}{4} + \frac{N}{2} - 1\}$. The lth subarray corresponds to the $(N^2/4 + N/2 - l + 1)$th to $(N^2 + N - l)$th rows of $\bar{\boldsymbol{v}}_i$, denoted as

$$\bar{\boldsymbol{v}}_i^l = \bar{\boldsymbol{A}}_i^l \boldsymbol{p}_i + \sigma_E^2 \boldsymbol{e}_l.$$

We can check that

$$\bar{\boldsymbol{v}}_i^l = \bar{\boldsymbol{A}}_i^1 \boldsymbol{\Phi}^{l-1} \boldsymbol{p}_i + \sigma_E^2 \boldsymbol{e}_l,$$

where $\boldsymbol{\Phi} = \mathrm{diag}(e^{-j(2\pi/\lambda)d\sin\theta_1}, e^{-j(2\pi/\lambda)d\sin\theta_2}, \ldots, e^{-j(2\pi/\lambda)d\sin\theta_K})$. Viewing $\bar{\boldsymbol{v}}_i^l$ as a newly received vector, we can get the equivalent covariance matrix $\boldsymbol{R}_i^l = \bar{\boldsymbol{v}}_i^l \bar{\boldsymbol{v}}_i^{lT}$. Taking the average of $\boldsymbol{R}_i^l$, we get

$$\boldsymbol{R}_i^{\mathrm{ave}} = \frac{1}{(\frac{N^2}{4} + \frac{N}{2})} \sum_{l=1}^{N^2/4+N/2} \boldsymbol{R}_i^l. \tag{10}$$

The spatially smoothed matrix $\boldsymbol{R}_i^{\mathrm{ave}}$ enables us to perform detection of $O(N^2)$ sources with N sensors.

B. Direction of Arrival Estimation

As mentioned in the introduction, we consider a narrowband decomposition for wideband DOA estimation. For each frequency component, we use the narrowband signal subspace method MUSIC to estimate.

Considering the spatial smoothing matrix $\boldsymbol{R}_i^{\mathrm{ave}}$ for ith frequency f_i, we do eigenvalue decomposition:

$$\mathrm{EVD}(\boldsymbol{R}_i^{\mathrm{ave}}) = \boldsymbol{U}_i \Lambda_i \boldsymbol{U}_i^T,$$

where $\Lambda_i = \mathrm{diag}(\lambda_i^1, \lambda_i^2, \ldots, \lambda_i^{N^2/4+N/2})$ are the eigenvalues and $\boldsymbol{U}_i = [\boldsymbol{u}_i^1, \boldsymbol{u}_i^2, \ldots, \boldsymbol{u}_i^{N^2/4+N/2}]$ is the corresponding eigenvector matrix. Suppose the eigenvalues are sorted decreasingly:

$$\lambda_i^1 \geq \lambda_i^2 \geq \ldots \geq \lambda_i^K > \lambda_i^{K+1} = \ldots = \lambda_i^{N^2/4+N/2}.$$

Then we can get the noise subspace $\boldsymbol{U}_i^E = [\boldsymbol{u}_i^{K+1}, \boldsymbol{u}_i^{K+2}, \ldots, \boldsymbol{u}_i^{N^2/4+N/2}]$, which consists of the last $N^2/4 + N/2 - K$ eigenvectors corresponding to the smallest $N^2/4 + N/2 - K$ eigenvalues. The estimated DOA can be found through an exhaustive search over all the direction space for the MUSIC spectrum:

$$M_i(\theta) = \frac{1}{(\boldsymbol{a}_i^\theta)^T \boldsymbol{U}_i^E (\boldsymbol{U}_i^E)^T \boldsymbol{a}_i^\theta},$$

where $\boldsymbol{a}_i^\theta = [1, a_i^\theta, \ldots, (\boldsymbol{a}_i^\theta)^{N^2/4+N/2-1}]$, with $a_i^\theta = e^{-j2\pi(f_c+f_i)d_I \sin(\theta)/c}$. Combining the resulting measurements for all the different frequencies, we construct the new combined MUSIC spectrum:

$$M(\theta) = \frac{1}{\frac{1}{I}\sum_{i=1}^{I} (\boldsymbol{a}_i^\theta)^T \boldsymbol{U}_i^E (\boldsymbol{U}_i^E)^T \boldsymbol{a}_i^\theta}. \tag{11}$$

Then the estimated DOAs are corresponding to the K largest values of the spectrum $M(\theta)$.

C. Wideband DOA Estimation Algorithm Based on MUSIC

According to section II, our observed data is $\boldsymbol{x}(t)$ in (7), and our problem of interest is to estimate the DOAs from the Fourier coefficients $\boldsymbol{y}(i),\ i = 1, \ldots, I$ in (8). The main difference with respect to the narrowband problem is that now the steering matrix $\boldsymbol{A}_i$ depends on the frequency component index i.

Suppose our total observation time is T_0, and we divide it into Q segments, with each segment $t_0 = t_2 - t_1$. We assume that there are I samples within each segment. Therefore, we have $I \cdot Q$ samples

$$\hat{\boldsymbol{X}} = [\hat{\boldsymbol{x}}(1), \hat{\boldsymbol{x}}(2), \ldots, \hat{\boldsymbol{x}}(I \cdot Q)]_{N \times (I \cdot Q)}$$

. For each segment q, we employ DFT to get the $N \times I$ corresponding frequency coefficient matrix:

$$\hat{\boldsymbol{Y}}_q = [\hat{\boldsymbol{y}}_q(1), \ldots, \hat{\boldsymbol{y}}_q(i), \ldots, \hat{\boldsymbol{y}}_q(I)],\ q = 1, \ldots, Q. \tag{12}$$

Considering all the segments, we can get the $N \times Q$ coefficient matrix for each frequency index i:

$$\hat{\boldsymbol{Y}}^i = [\hat{\boldsymbol{y}}_1(i), \ldots, \hat{\boldsymbol{y}}_q(i), \ldots, \hat{\boldsymbol{y}}_Q(i)],\ i = 1, \ldots, I. \tag{13}$$

The resulting sample covariance matrix for frequency index i can be written as

$$\hat{\boldsymbol{R}}_{\boldsymbol{y}_i} = \frac{1}{Q} \hat{\boldsymbol{Y}}^i (\hat{\boldsymbol{Y}}^i)^T. \tag{14}$$

Following the spatial smoothing technique in subsection A, we can get the sample spatial smoothing matrix $\hat{\boldsymbol{R}}_i^{\text{ave}}$. Accordingly, we can get the corresponding sample noise subspace $\hat{\boldsymbol{U}}_i^E$, and the combined MUSIC spectrum $\hat{M}(\theta)$. The algorithm is shown in Table I

TABLE I: Wideband DOA Estimation Algorithm

begin
- Obtain Q,I,$\hat{\boldsymbol{X}}$
- Obtain $\hat{\boldsymbol{Y}}_q,\ q = 1, \ldots, Q$ from $\hat{\boldsymbol{X}}$ via DFT
- i=1; %frequency index
- **do**
  - Obtain $\hat{\boldsymbol{Y}}^i$;
  - Obtain the covariance $\hat{\boldsymbol{R}}_{\boldsymbol{y}_i}$;
  - Obtain the spatial smoothing matrix $\hat{\boldsymbol{R}}_i^{\text{ave}}$;
  - Obtain the noise subspace $\hat{\boldsymbol{U}}_i^E$;
  - $i := i + 1$;
- **until** $i = I$;
- Calculate the combined MUSIC spectrum $\hat{M}(\theta)$ in (11);
- Estimate the DOAs as the K largest values of $\hat{M}(\theta)$.

end

IV. Numerical Examples

In this section, we use numerical examples to show the superior performance of our proposed strategy in terms of wideband source DOA estimation with a linear nested array.

In the examples, we consider a 2-level nested array with $N = 6$ sensors, with both the inner and outer ULAs having three sensors. The interspacing d_I is chosen as half of the shortest wavelength of the wideband signals, which ensures that there is no spatial aliasing. d_O is equal to $4d_I$. Suppose there are $K = 7$ wideband sources impinging from directions $\boldsymbol{\theta} = [-60^0, -35^0, -15^0, 5^0, 30^0, 45^0, 60^0]$. It is impossible for us to use a 6-ULA to detect seven sources. However the spatial smoothing matrix $\hat{\boldsymbol{R}}_i^{\text{ave}}$ in (10) helps a nested array obtain this goal. Suppose the wideband sources have the same center frequency $f_c = 100$ Hz and the same bandwidth $B = 40$ Hz. Besides, the sources follow zero mean Gaussian random processes with equal power, independent of each other. The noises are white Gaussian, and uncorrelated with the sources. The demodulated data is sampled at a frequency of 300 Hz.

A. MUSIC Spectra

The array output is decomposed into $I = 41$ narrowband components via DFT. The selection of proper value of I will be explained next. We choose the segment number to be $Q = 100$. Therefore we use a total of $I \times Q = 4100$ samples. Fig. 2 shows the representative MUSIC spectra using the spatial smoothing technique, with respect to various angles at a SNR of 0 dB. We take the SNR as

$$\text{SNR} = 10\log_{10}\frac{\text{E}[S(f)^2]}{\text{E}[E(f)^2]}.$$

We can see that the proposed method can resolve the 7 sources sufficiently well.

B. MSE versus SNR

We consider the performance of our proposed method by studying the MSE of the angle estimates as a function of the SNR. Since the 2-level nested array has 6 sensors and 12 degrees of freedom, we also consider the corresponding

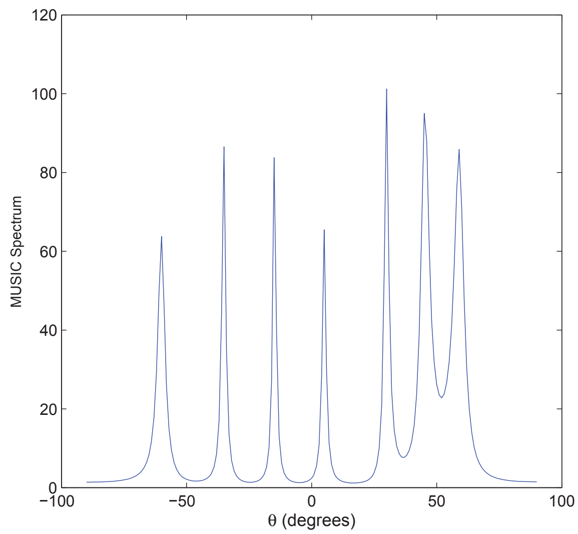

Fig. 2: MUSIC spectrum using the spatial smoothing technique, as a function of the DOA, $N = 6$, $K = 7$, $I = 41$, $Q = 100$, $\mathrm{SNR} = 0$ dB.

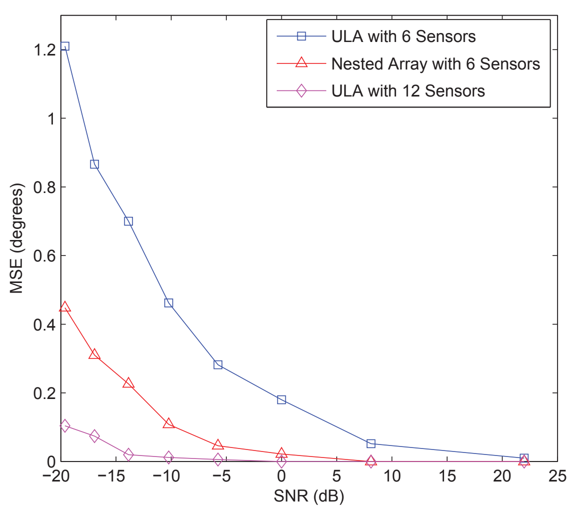

Fig. 3: MSE versus the SNR for the source at 30^0, using 2-level nested array, traditional ULA with 6 sensors and 12 sensors, $I = 41$,$Q = 100$.

MSE for conventional MUSIC applied to 6-element and 12-element ULAs. We plot the MSE for the source at 30^0. The performance is similar for the other sources. Fig. 3 shows the MSE of the three methods as a function of SNR, averaged over 500 Monte Carlo simulations, using $I \times Q = 4100$ samples.

We can see that the performance of all three methods improve with SNR. In addition, the nested array method performs reasonably better than the corresponding ULA with same number of sensors, and performs close to the much longer ULA with 12 sensors.

TABLE II: MSE versus different numbers of I

I	2	4	8	10	20
MSE	1.253	0.198	0.1	0.099	0.087
I	40	50	100	160	200
MSE	0.093	0.097	0.12	0.129	0.147

C. Impact of the choice of I

To investigate the impact on the estimation performance of the choice of I, we fixed the sample number at 4000. For different numbers of I, Table II shows the MSE results for estimation of a wideband source with $\theta = 30^0$. We can see that a moderate I guarantees good performance. When I is too small, it will lose information on most frequencies. On the other hand, when I is too large, the fusion process will perform badly.

V. Conclusion

In this paper, we proposed a novel approach for wideband source DOA estimation with the nested array. This approach can estimate more wideband sources than sensors, and obtain good estimation and resolution performance. Numerical examples demonstrate the effectiveness of our strategy. For future work, we will investigate the estimation performance for coherent wideband sources, and analyze source number detection using nested arrays.

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