Error Modeling and Calibration for High Resolution DOA Estimation

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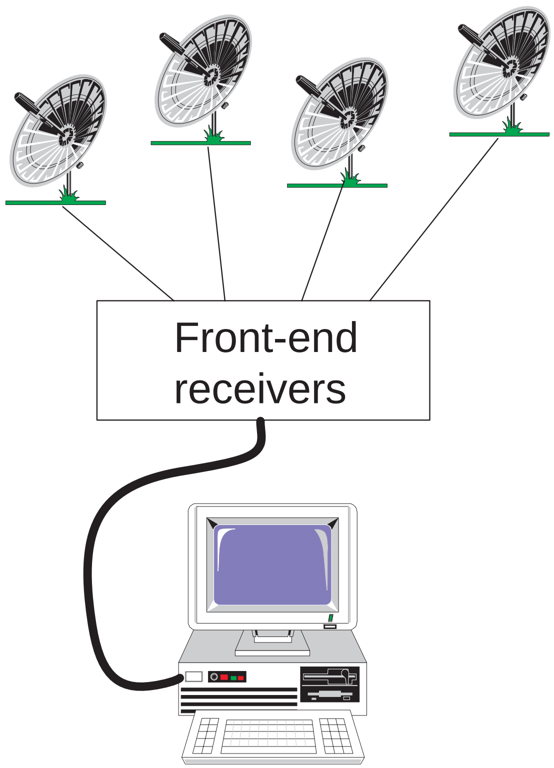
OUTLINE: DOA ESTIMATION

- **DOA Estimation Problem**
  - Problem formulation
  - Geometric data model
  - Some estimation methods
- Antenna Array Models and Errors
- Parametric Antenna Array Calibration
- Non-Parametric Antenna Array Calibration
- Application to DOA estimation
- Open Issues and Conclusions

DOA ESTIMATION PROBLEM

High-resolution Direction-of-Arrival (DOA) estimation using an array of sensors:



DATA MODEL

Baseband output of m sensors modeled by

$$\mathbf{x}(t) = \sum_{k=1}^{d} \mathbf{a}(\theta_k) s_k(t) + \mathbf{n}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t)$$

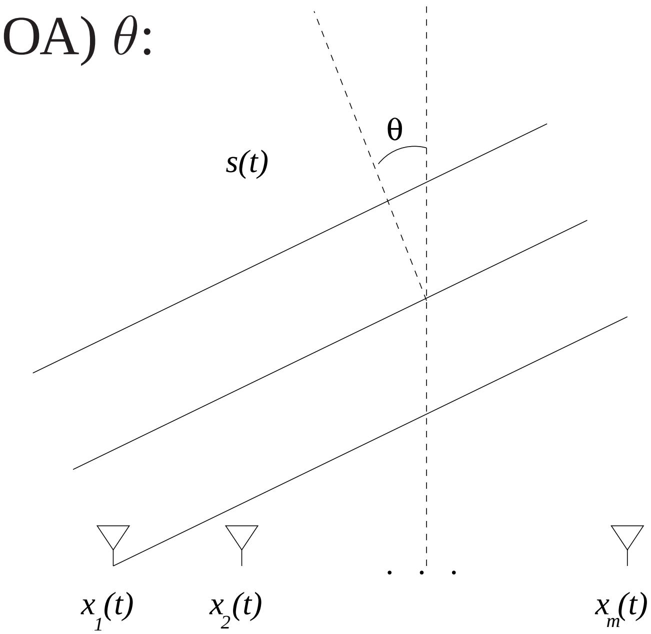
where

$\mathbf{x}(t) \in \mathcal{C}^m$	measurement at time t
$\mathbf{a}(\theta_k) \in \mathcal{C}^m$	array response in direction θ_k
$\theta_k \in \mathcal{R}$	DOA of kth source
$s_k(t) \in \mathcal{C}$	waveform of kth source

Given measurements $\{\mathbf{x}(t)\}_{t=1}^{N}$, desired to estimate d, $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_d]^T$ and $\mathbf{s}(t) = [s_1(t), \ldots, s_d(t)]^T$.

IDEAL ARRAY MANIFOLD

Plane wave, $s(t)$, arriving at a uniform linear array (ULA) from direction-of-arrival (DOA) θ:

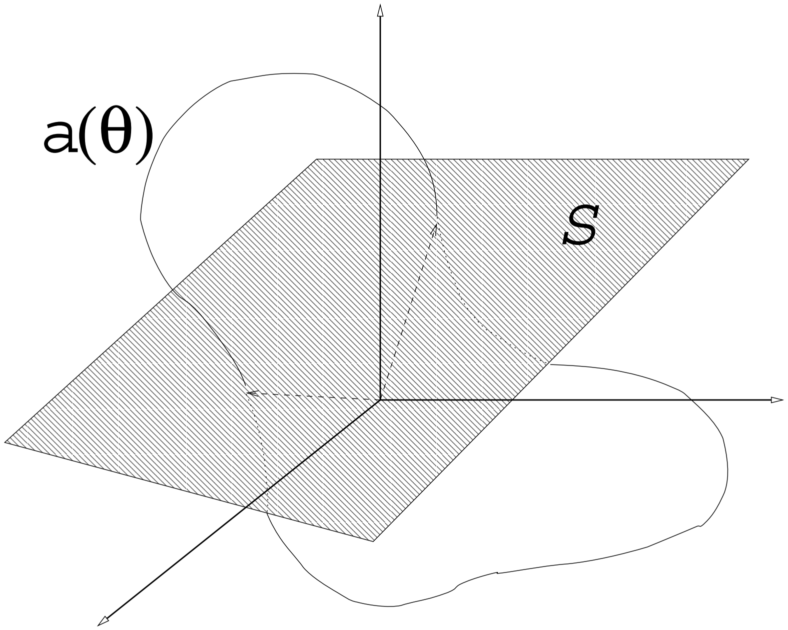


Ideal narrowband case: $x_l(t) = e^{-j\omega_c(l-1)(\Delta/c)\sin\theta} s(t),\ l = 1, \ldots, m$

Steering vector: $\mathbf{a}(\theta) = [1, e^{-j\omega_c(\Delta/c)\sin\theta}, \ldots, e^{-j\omega_c(m-1)(\Delta/c)\sin\theta}]^T$

ARRAY MANIFOLD

In general, steering vector $\mathbf{a}(\theta)$ is a "string" in $\mathcal{C}^m$-space:



Knowledge of array manifold $\{\mathbf{a}(\theta)\}_{\theta\in\Theta}$ is crucial for accurate DOA estimation. Requires **calibration**!

DOA ESTIMATION: BEAMFORMING

Classical approach to spatial filtering and DOA estimation!

Periodogram/Matched filter/Beamformer: $\hat{\theta}_k$ is the kth peak of the "spatial spectrum"

$$P(\theta) = \frac{1}{N} \sum_{t=1}^{N} |\mathbf{a}^*(\theta)\mathbf{x}(t)|^2$$

For a ULA:

$$\mathbf{a}(\theta) = [1, e^{j\phi}, \ldots, e^{j(m-1)\phi}]^T$$

where $\phi = \frac{\omega_c \Delta}{c} \sin\theta$ is the *electrical angle*. This is precisely the length-m periodogram, averaged over N realizations!

Simple and computationally efficient, but poor resolution: $\Delta\phi \approx 2\pi/m$

DOA ESTIMATION: NLLS/ML

Non-linear least-squares (=maximum likelihood for deterministic signals in white Gaussian noise):

$$\{\hat{\boldsymbol{\theta}}, \hat{\mathbf{S}}\} = \arg\min_{\boldsymbol{\theta},\mathbf{S}} \|\mathbf{X} - \mathbf{A}(\boldsymbol{\theta})\mathbf{S}\|_F^2$$

Separable problem: $\hat{\mathbf{S}}(\boldsymbol{\theta}) = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\mathbf{X}$, and

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \mathrm{Tr}\{\mathbf{P}_{\mathbf{A}}^{\perp}\hat{\mathbf{R}}\}$$

where $\mathbf{P}_{\mathbf{A}}^{\perp}$ is an orthogonal projection and $\hat{\mathbf{R}} = \frac{1}{N}\mathbf{X}\mathbf{X}^*$ is the sample covariance matrix.

NLLS has excellent performance but high complexity due to the non-linear optimization. Subspace methods are cheaper alternatives!

DOA ESTIMATION: MUSIC

If signal matrix $\mathbf{S}$ is full rank, noise-free data $\mathbf{X} = \mathbf{AS}$ spans *signal subspace:* $\mathrm{span}\{\mathbf{A}\}$.

Estimated from data by principal eigenvectors of array covariance:

$$\hat{\mathbf{R}} = \sum_{k=1}^{m} \hat{\lambda}_k \hat{\mathbf{e}}_k \hat{\mathbf{e}}_k^* = \hat{\mathbf{E}}_s \hat{\mathbf{\Lambda}}_s \hat{\mathbf{E}}_s^* + \hat{\mathbf{E}}_n \hat{\mathbf{\Lambda}}_n \hat{\mathbf{E}}_n^*$$

Ideally $\mathrm{span}\{\mathbf{E}_s\} = \mathrm{span}\{\mathbf{A}\}$, so $\mathrm{span}\{\mathbf{E}_n\} \perp \mathrm{span}\{\mathbf{A}\}$. Exploited by MUSIC method; "pseudo-spectrum":

$$P_{MU}(\theta) = \frac{1}{\|\hat{\mathbf{E}}_n^* \mathbf{a}(\theta)\|^2}$$

MUSIC DOA estimates are locations of d largest peaks of $P_{MU}(\theta)$.

Consistent and "cheap" estimates for large N and/or SNR!

OUTLINE: ARRAY MODELS

- DOA Estimation Problem
- **Antenna Array Models and Errors**
  - Physical antenna array modeling
  - Mutual coupling
  - Other error sources
- Parametric Antenna Array Calibration
- Non-Parametric Antenna Array Calibration
- Application to DOA estimation
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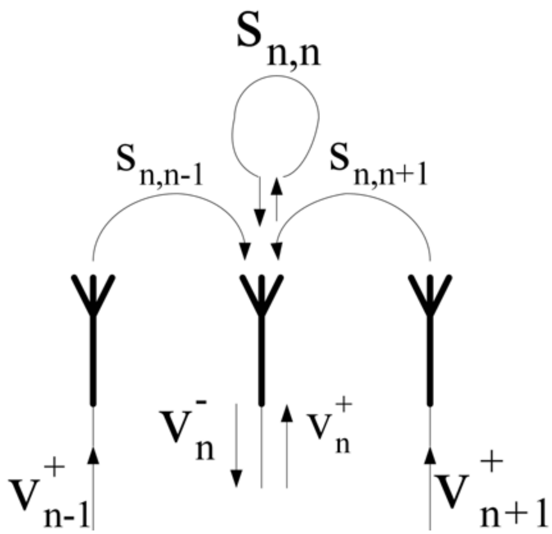
THE SCATTERING MATRIX

Microwave signals represented as traveling waves:



Given excitations $\{V_{n-1}^+, V_n^+, V_{n+1}^+\}$, the reflected wave at antenna n is

$$V_n^- = S_{n,n-1}V_{n-1}^+ + S_{n,n}V_n^+ + S_{n,n+1}V_{n+1}^+$$

The $S_{i,j}$ are the *scattering parameters* of the array!

ARRAY CHARACTERIZATION

Equivalent representations:

- Scattering matrix $\mathbf{S}$: $\mathbf{v}^- = \mathbf{S}\mathbf{v}^+$
- Mutual impedance matrix $\mathbf{Z}$: $\mathbf{j} = \mathbf{Z}\mathbf{v}$, where $\mathbf{j}$ are the currents and $\mathbf{v} = \mathbf{v}^+ + \mathbf{v}^-$ the gap voltages (for dipole antennas)
- Admittance matrix: $\mathbf{Y} = \mathbf{Z}^{-1}$

Scattering parameters are "easy" to measure. For an array of dipoles: $\mathbf{Z} = Z_c(\mathbf{I} - \mathbf{S})^{-1}(\mathbf{I} + \mathbf{S})$, where Z_c is the load impedance.

MUTUAL COUPLING

The received antenna voltages due to a far-field emitter V_s from DOA θ are proportional to:

$$\mathbf{v}^- \propto \mathbf{Ca}(\theta)V_s$$

where $\mathbf{a}(\theta)$ is the "geometric" steering vector and

$$\mathbf{C} = \mathbf{I} - \mathbf{S} \quad \text{(assuming identical and matched antennas)}$$

is the *mutual coupling* matrix.

Mutual coupling changes the steering vectors:

$$\mathbf{a}(\theta) \quad \rightarrow \quad \mathbf{a}_c(\theta) = \mathbf{Ca}(\theta)$$

SOURCES OF MODELING ERROR

In practice, array manifold not perfectly known:

- Mutual coupling unknown or mis-specified
- Antenna element positions and orientation not perfectly known
- Gain and phase imbalances among receivers
- IQ-imbalances in receivers
- Near-field scattering due to platform or terrain
- Etc...

Remedy: **array calibration**!

OUTLINE: PARAMETRIC CALIBRATION

- DOA Estimation Problem
- Antenna Array Models and Errors
- **Parametric Array Calibration**
  - Auto-calibration
  - Calibration using sources at known positions
- Non-Parametric Array Calibration
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AUTO-CALIBRATION

Parametric model of the antenna, including "array parameters" $\boldsymbol{\rho}$: $\mathbf{a}(\theta, \boldsymbol{\rho})$

Example: If only element positions (x_k, y_k) are unknown we can use

$$\mathbf{a}(\theta, [\mathbf{x}; \mathbf{y}]) = [e^{2\pi j(x_1 \cos\theta + y_1 \sin\theta)}, \ldots, e^{2\pi j(x_m \cos\theta + y_m \sin\theta)}]^T$$

Auto-calibration estimates all parameters simultaneously, e.g. using NLLS:

$$\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}}, \hat{\mathbf{S}}\} = \arg \min_{\boldsymbol{\theta}, \boldsymbol{\rho}, \mathbf{S}} \|\mathbf{X} - \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{S}\|_F^2$$

Ill-conditioned problem! If approximate values $\boldsymbol{\rho}_0$ are known, a Bayesian (MAP) approach is possible:

$$\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}}, \hat{\mathbf{S}}\} = \arg \min_{\boldsymbol{\theta}, \boldsymbol{\rho}, \mathbf{S}} \|\mathbf{X} - \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{S}\|_F^2 + \frac{1}{2}\|\boldsymbol{\rho} - \boldsymbol{\rho}_0\|_{\mathbf{C}_\rho^{-1}}^2$$

MAP can mitigate **small** perturbations for "general" parameterizations.

AUTO-CALIBRATION: LITERATURE

- Overview: Li, Gan and Ye, ISAP2003
- Array shape calibration: Rockah and Schultheiss, Trans ASSP 1987; Weiss and Friedlander, Trans ASSP 1989; Wan et al, OCEANS2001; Park et al, OCEANS2001& OCEANS2004
- Gain and phase calibration: Paulraj and Kailath, ICASSP85; Astély et al, Trans SP 1999
- Mutual coupling: Friedlander and Weiss, Trans AP 1991; Solomon et al, ICASSP 1998; Jaffer, IEEE Radar 2002
- Partially calibrated arrays: Swindlehurst, ICASSP95; Weiss and Friedlander, Trans AES 1996; See and Gershman, Trans SP 2004
- General perturbations, using prior information: Viberg and Swindlehurst, Trans SP 1994; Jansson et al, Trans SP 1998

CALIBRATION MEASUREMENTS

Calibration data collected with sources at known positions $\{\theta_c\}_{c=1}^K$:

$$\mathbf{x}_c(t) = \mathbf{a}_c s_c(t) + \mathbf{n}_c(t)\,,\ t = 1,\ldots,N;\ c = 1,\ldots,K$$

Coherent calibration ($s_c(t)$ known):

$$\hat{\mathbf{a}}_c = \frac{\sum_{t=1}^N \mathbf{x}_c(t) s_c^*(t)}{\sum_{t=1}^N |s_c(t)|^2}$$

Non-coherent calibration: principal eigenvector of sample covariance

$$\hat{\mathbf{R}}_c = \frac{1}{N}\sum_{t=1}^N \mathbf{x}_c(t)\mathbf{x}_c^*(t) = \sum_{k=1}^m \lambda_k \mathbf{e}_k \mathbf{e}_k^* \ \Rightarrow\ \hat{\mathbf{a}}_c \propto \mathbf{e}_1$$

Data without sources can be used to determine noise color.

PARAMETRIC CALIBRATION

If a model is known, the estimated steering vectors $\{\hat{\mathbf{a}}_c\}_{c=1}^K$ (from calibration data) can be used to estimate the array parameters:

$$\hat{\boldsymbol{\rho}} = \arg\min_{\hat{\boldsymbol{\rho}}} \sum_{c=1}^{K} \|\hat{\mathbf{a}}_c - \mathbf{a}(\theta_c, \boldsymbol{\rho})\|^2$$

(coherent calibration), or

$$\{\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\gamma}}\} = \arg\min_{\boldsymbol{\rho},\boldsymbol{\gamma}} \sum_{c=1}^{K} \|\hat{\mathbf{a}}_c - \mathbf{a}(\theta_c, \boldsymbol{\rho})\gamma_c\|^2$$

(non-coherent calibration)

Some publications:

- Algorithms and performance analysis: See, IEE Proc 1995; Ng and Nehorai, Signal Processing 1995; Ng and See Trans AP 1996

OUTLINE: NON-PARAMETRIC CALIBRATION

- DOA Estimation Problem
- Antenna Array Models and Errors
- Parametric Array Calibration
- **Non-Parametric Array Calibration**
  - Global calibration
  - Array interpolation
  - Local array interpolation
- Application to DOA Estimation
- Open Issues and Conclusions

NON-PARAMETRIC CALIBRATION

Global calibration (e.g. Pierre and Kaveh, DSP 1995):

$$\mathbf{a}_c(\theta) = \mathbf{Q}\mathbf{a}(\theta)$$

- $\mathbf{a}_c(\theta)$ desired "true" array manifold, measured for $\{\theta_c\}_{c=1}^{K}$
- $\mathbf{Q}$ diagonal matrix: gain+phase errors only
- $\mathbf{Q}$ full matrix: gain+phase+mutual coupling

Correction matrix determined using least-squares:

$$\hat{\mathbf{Q}} = \arg\min_{\mathbf{Q}} \|\hat{\mathbf{A}}_c - \mathbf{Q}\mathbf{A}(\boldsymbol{\theta}_c)\|_F^2$$

Simple and efficient, but cannot handle DOA-dependent errors!

ARRAY INTERPOLATION

Manifold Separation Theorem: (Doron and Doron, Trans SP 1994)
Under mild conditions, any array can be represented by

$$\mathbf{a}_c(\theta) = \mathbf{M}\mathbf{v}(\theta) + \mathbf{r}_M(\theta)$$

where $\mathbf{M}$ is $m \times M,\ M \geq m$ and

$$\mathbf{v}(\theta) = [1, e^{j\theta}, \ldots, e^{j(M-1)\theta}]^T$$

The error $\|\mathbf{r}_M(\theta)\|$ vanishes "rapidly" as $M \to \infty$.

Given cal data, "sampling matrix" determined by

$$\hat{\mathbf{M}} = \arg\min_{\mathbf{M}} \|\hat{\mathbf{A}}_c - \mathbf{M}\mathbf{V}(\boldsymbol{\theta}_c)\|_F^2$$

Can give accurate global interpolation, but requires dense and accurate cal data!

LOCAL ARRAY INTERPOLATION

Obvious approach: interpolate cal data $\{\hat{\mathbf{a}}_c(\theta_c)\}_{c=1}^{K}$ using linear or spline interpolation. But $\mathbf{a}_c(\theta)$ may not be smooth!

Better approach: exploit *nominal model* $\mathbf{a}(\theta)$. Express true array manifold as:

$$\mathbf{a}_c(\theta) = \mathbf{Q}(\theta)\mathbf{a}(\theta)$$

where $\mathbf{Q}(\theta) = \mathrm{diag}\{\mathbf{q}(\theta)\}$ is a local (θ-dependent) *correction matrix*.

Using cal data, $\mathbf{q}(\theta)$ can be measured at $\{\theta_c\}_{c=1}^{K}$:

$$\hat{\mathbf{q}}(\theta_c) = \hat{\mathbf{a}}_c./\mathbf{a}(\theta_c)$$

Idea: interpolate $\mathbf{q}(\theta)$ instead, normally much smoother than $\mathbf{a}_c(\theta)$!

Linear or spline interpolation OK, but again requires dense and accurate cal.

WEIGHTED REGRESSION

Simple "machine learning" approach: interpolate kth correction factor by

$$\hat{q}_k(\theta) = \frac{1}{\sum_{c=1}^{K} w(\theta - \theta_c)} \sum_{c=1}^{K} w(\theta - \theta_c)\hat{q}_k(\theta_c)\,,\ k = 1, \ldots, m$$

where $w(x)$ is a *Kernel function*, e.g. $w(x) = e^{-x^2/h^2}$.

One can treat real and imaginary parts separately, or whole vector $\mathbf{q}(\theta)$ in one shot (manifold learning).

Critical parameter: Kernel bandwidth h (possibly θ-dependent). Trades bias for variance. Can use Cross-Validation (CV) or "Intersection of Confidence Intervals" (ICI) (e.g. Katkovnic, SP Letters 1999).

LOCAL POLYNOMIAL MODELING

Alternative approach: express cal data using basis functions $\{\phi_l(x)\}_{l=0}^{p}$, centered at "query point" θ:

$$\hat{q}_k(\theta_c) = \sum_{l=0}^{p} \alpha_l \phi_l(\theta_c - \theta) = \boldsymbol{\phi}^T(\theta_c - \theta)\boldsymbol{\alpha}$$

Usually $\phi_l(x) = x^l$ and $p = 1$ (locally linear interpolation).

Polynomial coefficients determined by weighted regression:

$$\hat{\boldsymbol{\alpha}} = \arg\min_{\alpha} \sum_{c=1}^{K} w(\theta_c - \theta) \left|\hat{q}_k(\theta_c) - \boldsymbol{\phi}^T(\theta_c - \theta)\boldsymbol{\alpha}\right|^2$$

The interpolated value is then simply $\hat{q}_k(\theta) = \hat{\alpha}_0$.

Bandwidth of Kernel $w(x)$ again chosen by CV or ICI (Bootstrap?)

NON-PARAMETRIC CAL: LITERATURE

- Global cal and experimental validation: Pierre and Kaveh, DSP 1995; Dandekar et al, ICPWC 2000; Gupta et al, Trans AP 2003; Pettersson and Grahn, Phased Array Symp 2003
- Manifold sep: Hung and Wu, SPIE 1991; Doron and Doron, Trans SP 1994 (Parts I-III); Weiss and Friedlander, IEE Proc 1994; Terrés et al, VTC02; Bühren et al, ICASSP03; Belloni et al, Trans SP 2007
- Local array interpolation: Ottersten and Viberg, Asilomar91; Lanne et al, Asilomar06; Lundgren et al, ICASSP07
- General local approximation: Katkovnic, SP Letters 1999; Bose and Ahuja, Trans IP 2006 (Moving Least-Squares)
- Yet an alternative approach - interpolate using "spectrum": Wu, ICASSP91; Schmidt, Trans SP 1992; Xue et al, APS98

OUTLINE: APPLICATION

- DOA Estimation Problem
- Antenna Array Models and Errors
- Parametric Array Calibration
- Non-Parametric Array Calibration
- **Application to DOA Estimation**
  - Simulation setup
  - Results
- Open Issues and Conclusions

SIMULATION SETUP

A simple experiment to compare calibration approaches:

- Nominal array: $m = 10$ element ULA
- Two i.i.d. sources at $\theta_1 = 10°$, $\theta_2 = 15°$ w.r.t. broadside
- Perturbations:
  - Position errors: $(\Delta x_k, \Delta y_k) \sim \mathcal{N}(0, 10\%)$
  - Gain and phase errors: $\mathcal{N}(0, 10\%)$
- No finite sample effects ($N = \infty$)
- Calibration data collected at DOAs $\theta_c \in \{-40° : 5° : 40°\}$ ($K = 17$)
- Calibration data: $\hat{\mathbf{a}}_c \sim \mathcal{N}(\mathbf{a}_c, 0.01^2\mathbf{I})$

SIMULATION SETUP: METHODS

The MUSIC algorithm is used for DOA estimation, together with:

- No calibration (reference)
- Global calibration with diagonal $\mathbf{Q}$
- Global calibration with full $\mathbf{Q}$
- Calibration by estimating element positions
- Linear interpolation of $\hat{\mathbf{a}}_c$
- Linear interpolation of $\hat{\mathbf{q}}_c = \hat{\mathbf{a}}_c./\mathbf{a}(\theta_c)$
- Locally linear approximation (optimal bandwidth)

SIMULATION RESULTS

DOA estimation statistics based on 100 Monte-Carlo runs.

RMS error of $\hat{\theta}_1$:

No Cal	Diag $\mathbf{Q}$	Full $\mathbf{Q}$	Pos	$\hat{\mathbf{a}}_c$-int	$\hat{\mathbf{q}}_c$-int	Local
$4.73°$	$0.16°$	$0.05°$	$3.15°$	$10.10°$	$0.05°$	$\mathbf{0.02}°$

RMS error of $\hat{\theta}_2$:

No Cal	Diag $\mathbf{Q}$	Full $\mathbf{Q}$	Pos	$\hat{\mathbf{a}}_c$-int	$\hat{\mathbf{q}}_c$-int	Local
$1.24°$	$0.23°$	$0.04°$	$0.61°$	$2.25°$	$0.05°$	$\mathbf{0.02}°$

OUTLINE: CONCLUSIONS

- DOA Estimation Problem
- Antenna Array Models and Errors
- Parametric Array Calibration
- Non-Parametric Array Calibration
- Application to DOA Estimation
- **Open Issues and Conclusions**

OPEN ISSUES

- A more thorough performance comparison
- Improved calibration techniques
  - Combining parametric and non-parametric?
  - Combining global and local methods?
  - Tracking changes in array manifold
- Design issues for (local) interpolation:
  - Choice of basis functions
  - Choice of bandwidth
- Statistical performance analysis (known scenario)
- Tools for predicting performance from data only
  - How can Boostrap be used?

CONCLUDING REMARKS

- Any real-world application of DOA estimation requires calibration
- There are no perfect models, even using "array parameters"
- Calibration using sources at known positions is very effective, especially non-parametric techniques
- Interpolate a *correction* to a nominal model, rather than the cal data directly
- Local modeling is a promising (but expensive!) approach, but requires more research
- **Last note:** also hardware calibration necessary - DSP cannot recover energy that is lost due to mismatch!