Capacity of Underspread Fading Channels

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joint work with

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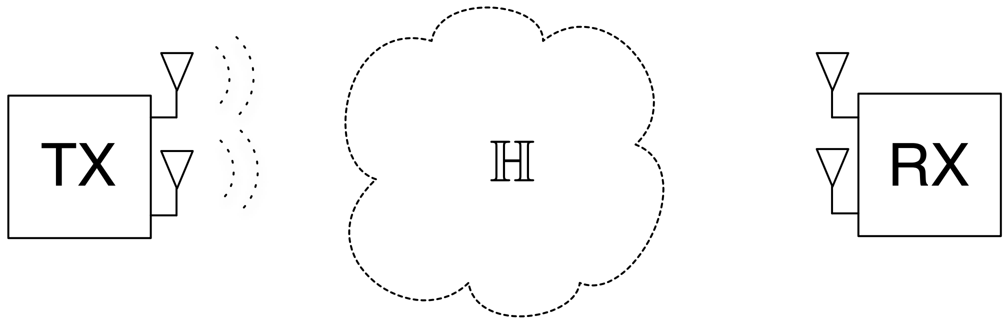
A General Model for Wireless Systems



- The **continuous-time** channel $\mathbb{H}$ is random, dispersive in both **time** and **frequency**, and correlated in **space**

- Neither TX nor RX know the realizations of $\mathbb{H}$ but the channel law is known to both TX and RX (**noncoherent** setting)

- **Peak** and **average power** of the TX signal are constrained

Fundamental Tradeoffs

Noncoherent capacity is the **ultimate limit** on the rate of reliable communication over fading channels

- It reveals a fundamental tradeoff between **degrees of freedom** (i.e., dimensions) **in signal space** and **channel uncertainty**

- It sheds light on relevant **design questions/issues**:
  - How much **bandwidth** and how many **antennas** to use?
  - Impact of **propagation conditions** (e.g., delay spread, Doppler spread, spatial correlation) on system performance?
  - **Difference** between **coherent** (i.e., genie-aided) and **noncoherent** capacity

A Brief Literature Survey

Many (mostly asymptotic) results are available in the literature

- for **discretized** and/or **discrete-time** channel models
- under different assumptions on
  - the **channel model**: block-fading, time-selective only, frequency-selective only, basis expansion model
  - the **input signal**: average power constraint, peak constraint in time, in time & frequency, in time & frequency & space

A Brief Literature Survey (Cont'd)

SISO case:

- The capacity of WSSUS Rayleigh fading channels **equals** the **AWGN capacity** in the wideband limit [*Gallager, 1968; Kennedy, 1969*]

- **Peaky signals** are needed to achieve the wideband AWGN capacity [*Gallager, 1968; Telatar & Tse, 2000; Verdú, 2002*]

- Under **peak constraints**, AWGN capacity **cannot** be achieved [*Viterbi, 1967; Médard & Gallager, 2002; Subramanian & Hajek, 2002*]

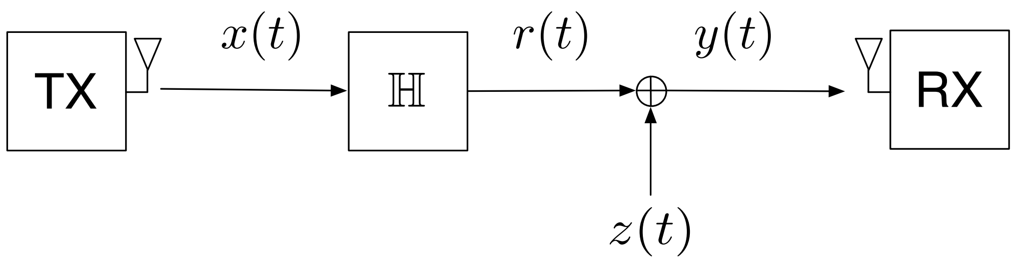
A Brief Literature Survey (Cont'd)

MIMO case:

- Low-SNR **bounds** on capacity can be found in [*Liang & Veeravalli, 2004; Borgmann & HB, 2005; Srinivasan & Varanasi, 2007; Sethuraman et al., 2008*]

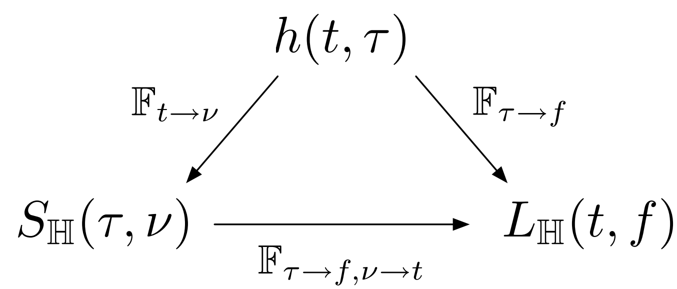
- Spatial correlation is **beneficial** in the noncoherent setting [*Jafar & Goldsmith, 2005; Srinivasan & Varanasi, 2007; Zhang & Laneman, 2007*]

Transfer Function Calculus for LTV Channels



$$r(t) = \int_{\tau} h(t,\tau)x(t-\tau)d\tau = \iint_{\tau\,\nu} S_{\mathbb{H}}(\tau,\nu)x(t-\tau)e^{j2\pi\nu t}d\tau d\nu$$

- $S_{\mathbb{H}}(\tau,\nu)$: **delay-Doppler** spreading function
- $L_{\mathbb{H}}(t,f)$: time-varying **transfer function**



WSSUS and Underspread Assumptions

- We model the channel as a **WSSUS Gaussian** random process

$$\mathbb{E}[S_{\mathbb{H}}(\tau,\nu)] = 0$$
$$\mathbb{E}[S_{\mathbb{H}}(\tau,\nu)S_{\mathbb{H}}^{*}(\tau',\nu')] = C_{\mathbb{H}}(\tau,\nu)\delta(\tau-\tau')\delta(\nu-\nu')$$

$C_{\mathbb{H}}(\tau,\nu)$: **scattering function**

- $\mathbb{H}$ is said to be **underspread** if $C_{\mathbb{H}}(\tau,\nu)$ is **highly concentrated** in the τ–ν plane

- We assume $C_{\mathbb{H}}(\tau,\nu)$ to be supported within the **rectangle** $[-\tau_0,\tau_0]\times[-\nu_0,\nu_0]$ of area $\Delta_{\mathbb{H}} = 4\tau_0\nu_0 \ll 1$

- Wireless channels are **highly underspread**, with $\Delta_{\mathbb{H}} \in [10^{-7}, 10^{-3}]$

Results for the General Continuous-Time WSSUS Model

When the peak power of the TX signal is constrained in time and frequency:

- Noncoherent capacity **approaches zero** as bandwidth becomes large [*Médard & Gallager, 2002; Subramanian & Hajek, 2002*]

- A **nonasymptotic upper bound** on the rate achievable with constant modulus constellations over **underspread** channels was obtained in [*Schafhuber et al., 2004*]

- The upper bound in [*Schafhuber et al., 2004*] is explicit in the channel's **scattering function** and hints at the existence of a **capacity-maximizing** bandwidth

Scope of This Talk

- We analyze the **general class** of continuous-time **underspread** (MIMO) channels under a peak constraint in time & frequency
- Contributions:
  - **Upper** and **lower bounds** on capacity, explicit in the channel's **scattering function** and the number of antennas
  - Possible to identify the **capacity-optimal** combination of bandwidth and antennas
  - **Exact expression** for the **first-order Taylor series expansion** of capacity in the infinite-bandwidth limit
  - Spatial correlation is **beneficial** in the wideband regime

Discretization Through Eigenvalue Decomposition

$$y(t) = \int_{\tau} h(t, \tau)x(t - \tau)d\tau + z(t)$$

- Almost all tools for information-theoretic analysis require a **discretized representation** of the I/O relation

- A classic approach is to transmit and receive on the channel's **eigenfunctions** $\Rightarrow$ **countable set** of **scalar** I/O relations

- This approach has been successfully used to compute:
  - The capacity of a **bandlimited AWGN** channel [*Wyner, 1966*]
  - The capacity of a **deterministic LTI** channel [*Gallager, 1968*]

Eigenfunctions of Random Channels

- The eigenfunctions of random LTI channels are **complex sinusoids**, irrespectively of the realization of the impulse response $h(t)$
- In the LTV case, the eigenfunctions are, in general, **random** and **not known** to TX and RX in the noncoherent setting
- The eigenfunctions of **underspread** LTV channels can be well approximated by a set of **deterministic** functions [*Kozek, 1997*]
- The **calculus** for underspread LTV channels is **essentially identical** to that for LTI channels

Diagonalization of Underspread Channels

- Underspread channels can be **approximately diagonalized** as:

$$h(t, t-t') \approx \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} L_{\mathbb{H}}(kT, nF) g_{k,n}(t) g_{k,n}^{*}(t')$$

- $\left\{g_{k,n}(t) = g(t-kT)e^{j2\pi nFt}\right\}$ is an **orthonormal Weyl-Heisenberg** set

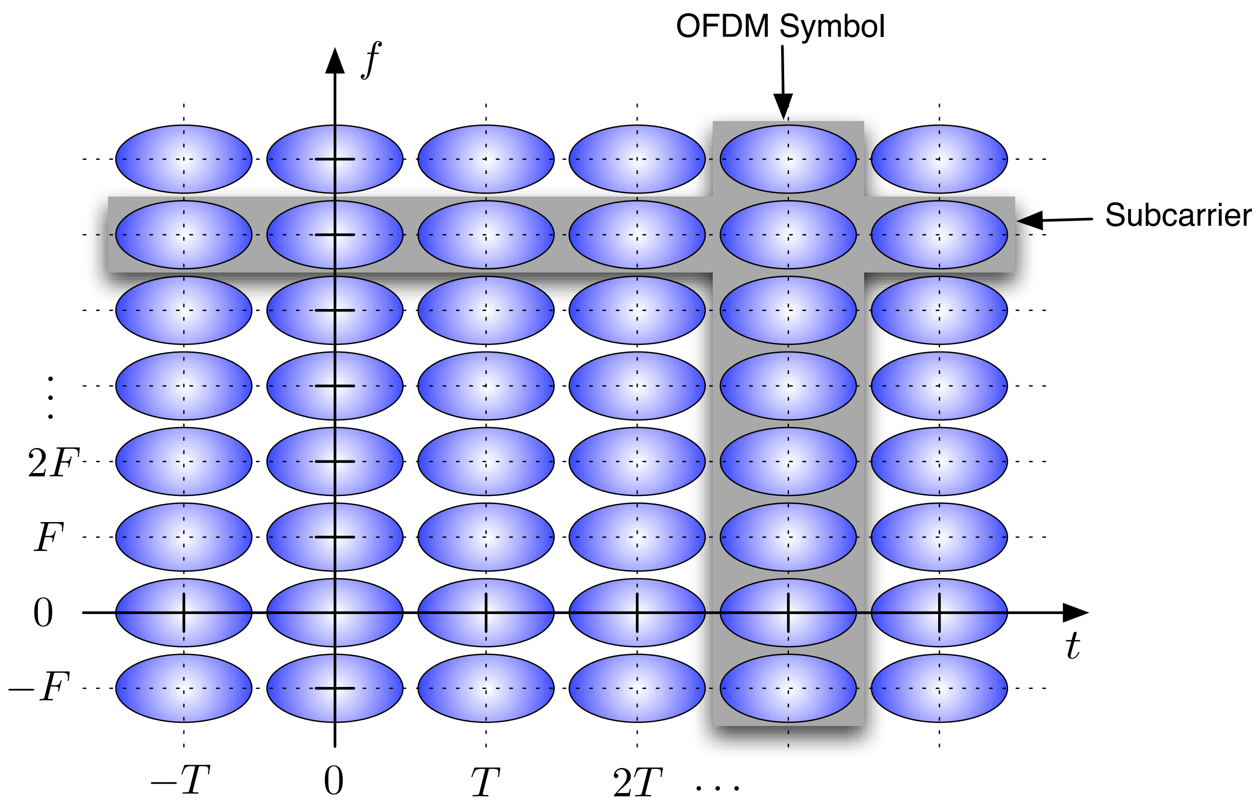
- The prototype function $g(t)$ is **well localized** in time and frequency

- The grid parameters T and F satisfy $1 < TF \leq 1/\Delta_{\mathbb{H}}$

- Projection of $y(t)$ and $x(t)$ onto $\{g_{k,n}(t)\}$ yields **discretized** I/O relation

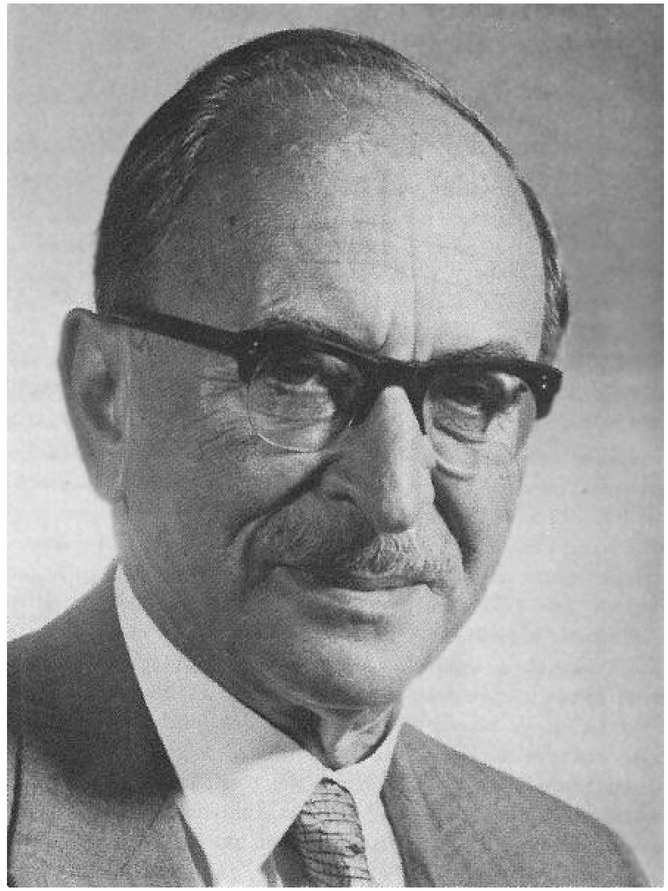
$$y[k,n] = \underbrace{L_{\mathbb{H}}(kT, nF)}_{h[k,n]} x[k,n] + z[k,n]$$

Transmission on Channel's Eigenfunctions: Pulse-Shaped OFDM



Implemented as **pulse-shaped OFDM:** $x(t) = \sum_{k,n} x[k,n] g(t-kT) e^{j2\pi nFt}$

The Inventor of OFDM



D. Gabor, 1900-1979
Nobel Prize in Physics (1971) for inventing holography

Power Constraints

- One **channel use** takes place over K OFDM symbols and N subcarriers
- The **vector of symbols** transmitted in a channel use is denoted as $\mathbf{x}$
- **Average-power constraint**:

$$\mathbb{E}\left[\|\mathbf{x}\|^2\right]/T \leq KP$$

- Peak constraint in **time** and **frequency**:

$$\frac{1}{T}\left|x[k,n]\right|^2 \leq \frac{\beta P}{N}, \qquad \text{a.s.,} \qquad \text{for all } k,n$$

- $\beta \geq 1$ is the nominal **peak- to average-power ratio**

Capacity Bounds

- Capacity as a function of bandwidth $W = NF$ is defined as

$$C(W) = \lim_{K\to\infty} \frac{1}{KT} \sup_{\mathcal{P}_{\mathbf{x}}} I(\mathbf{y};\mathbf{x}) \qquad \text{[bit/s]}$$

- $C(W)$ is **difficult** to compute for general W

- We obtain **upper** and **lower** bounds that are explicit in the channel's scattering function

Upper Bound

Theorem [*Durisi et al., 2008*]: The capacity of an underspread WSSUS Rayleigh fading channel is upper-bounded as

$$C(W) \leq \mathrm{U}(W) = \frac{W}{TF} \log\left(1 + \alpha(W)\frac{PTF}{W}\right) - \alpha(W)A(W)$$

with

$$\alpha(W) = \min\left\{1, \frac{W}{TF}\left(\frac{1}{A(W)} - \frac{1}{P}\right)\right\}$$

and

$$A(W) = \frac{W}{\beta} \iint\limits_{\tau\,\nu} \log\left(1 + \frac{\beta P}{W} C_{\mathbb{H}}(\tau, \nu)\right) d\nu d\tau$$

Remarks on the Upper Bound

- For all wireless channels and SNR values of **practical interest**

$$\mathrm{U}(W) = \frac{W}{TF}\log\left(1 + \frac{PTF}{W}\right) - \frac{W}{\beta}\int_\tau\int_\nu \log\left(1 + \frac{\beta P}{W}C_{\mathbb{H}}(\tau,\nu)\right) d\nu d\tau$$

- The **first** term is the capacity of an **AWGN channel** with power P and W/TF degrees of freedom

- The **second** term is a **penalty** term due to channel uncertainty

- $\mathrm{U}(W) \to 0$ as $W \to \infty$

Lower Bound

Theorem [*Durisi et al., 2008*]: The capacity of an underspread WSSUS Rayleigh fading channel is lower-bounded as $C(W) \geq \mathrm{L}(W)$, where

$$\mathrm{L}(W) = \max_{1 \leq \gamma \leq \beta} \left\{ \frac{W}{\gamma T F} I(y; x \mid h) - \frac{1}{\gamma T} \int_{-1/2}^{1/2} \log\det\left(\mathbf{I}_N + \frac{\gamma P T F}{W} \mathbf{C}(\theta) \right) d\theta \right\}$$

- $I(y; x \mid h)$: **coherent** mutual information of the scalar channel $y = hx + z$, where $h, z \sim \mathcal{CN}(0, 1)$
- x: zero mean, **constant modulus**, with $|x|^2 = \gamma PT/N$
- $\mathbf{C}(\theta)$: matrix-valued **spectral density** of the multivariate random process $\left\{ \mathbf{h}[k] = \left[h[k, 0] \; h[k, 1] \; \cdots \; h[k, N-1] \right]^T \right\}_{k \in \mathbb{Z}}$

Remarks on the Lower Bound

- Standard results on the **asymptotic equivalence** between circulant and Toeplitz matrices [*Pearl, 1973*] lead to a **looser lower bound** that is **explicit** in $C_{\mathbb{H}}(\tau, \nu)$

- For large bandwidth the lower bound takes a **simple** form

$$
\begin{aligned}
\mathrm{L}(W) &\approx \mathrm{L_a}(W) \\
&= \max_{1 \le \gamma \le \beta} \left\{ P - \frac{\gamma P^2 T F}{W} - \frac{W}{\gamma} \iint\limits_{\tau\,\nu} \log\left(1 + \frac{\gamma P}{W} C_{\mathbb{H}}(\tau, \nu)\right) d\nu d\tau \right\}
\end{aligned}
$$

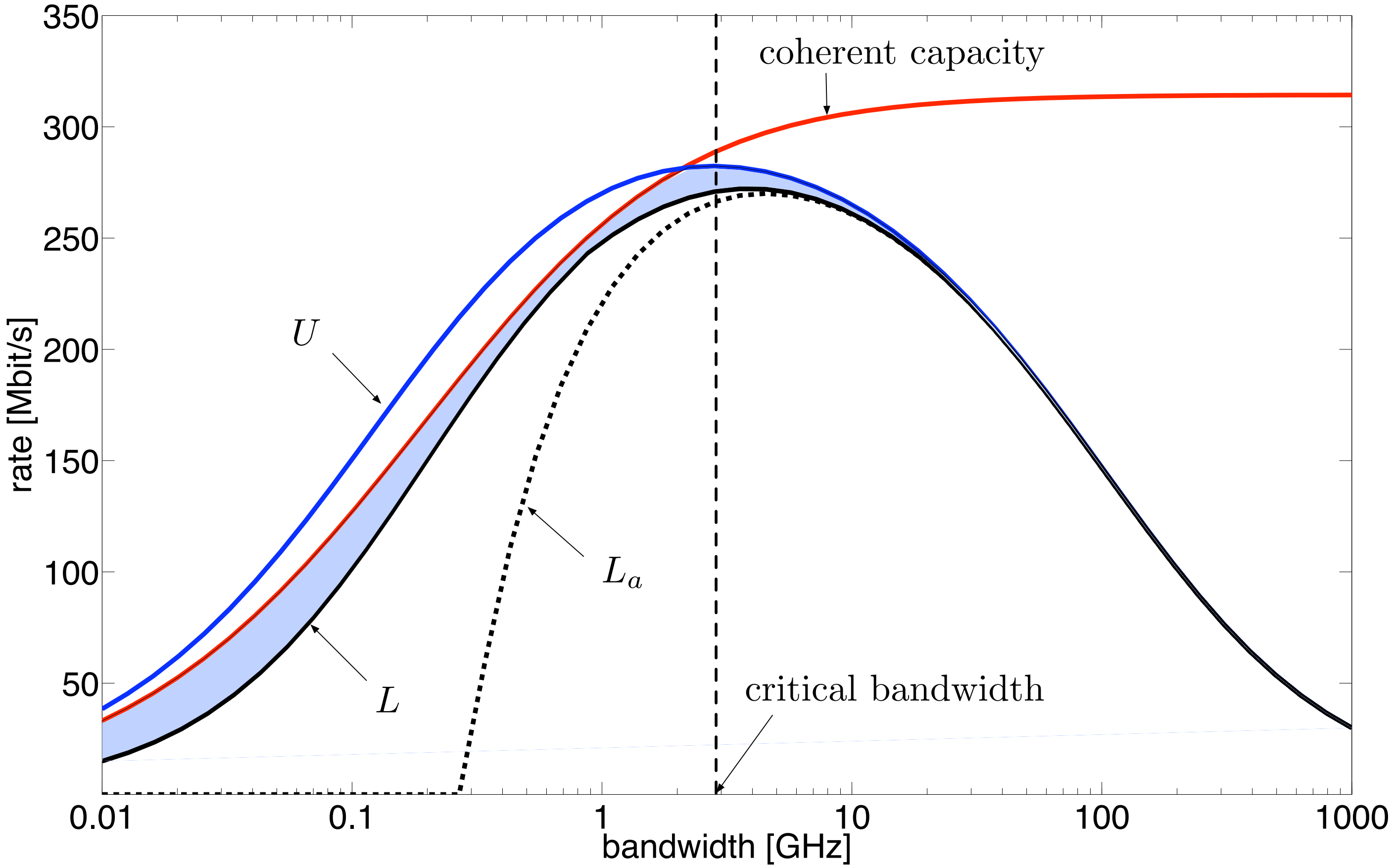
Key Elements of the Proofs

Use the chain rule for mutual information to obtain

$$I(\mathbf{y};\mathbf{x}) = I(\mathbf{y};\mathbf{x},\mathbf{h}) - I(\mathbf{y};\mathbf{h}\,|\,\mathbf{x})$$

- For the upper bound also use
  - First term: Gaussian distribution is differential-entropy maximizer
  - Second term: **relation** between **mutual information** and **MMSE** according to [*Guo et al., 2005*]
- For the lower bound
  - First term: $I(\mathbf{y};\mathbf{x},\mathbf{h}) \geq I(\mathbf{y};\mathbf{x}\,|\,\mathbf{h})$
  - Second term: a generalization of Szegö's theorem to **block-Toeplitz** matrices with **Toeplitz blocks** [*Miranda & Tilli, 2000*]

Numerical Evaluation of the Bounds for System Parameters Relevant to IEEE802.11a, $\Delta_{\mathbb{H}} = 10^{-3}$



Bounds Shed Light on Relevant Design Issues

- $\mathrm{U}(W)$ and $\mathrm{L}(W)$ take on their maximum at **large** but **finite** bandwidth W
- Beyond this **critical bandwidth**, the use of additional bandwidth is **detrimental**
- Many **current** wireless systems operate **well below** the critical bandwidth

The Large-Bandwidth Regime

Theorem [*Durisi et al., 2008*]: The capacity of an underspread WSSUS Rayleigh fading channel satisfies

$$\lim_{W\to\infty} WC(W) = \begin{cases} \frac{P^2}{2}\left(\beta\kappa_{\mathbb{H}} - TF\right), & \text{if } \beta > \frac{2TF}{\kappa_{\mathbb{H}}} \\ \frac{(\beta P\kappa_{\mathbb{H}})^2}{8TF}, & \text{if } \beta \leq \frac{2TF}{\kappa_{\mathbb{H}}} \end{cases}$$

where

$$\kappa_{\mathbb{H}} = \int_{\tau}\int_{\nu} C^2_{\mathbb{H}}(\tau,\nu)d\nu d\tau$$

- **First-order Taylor series** expansion of $C(W)$ around $1/W = 0$ fully characterized
- $\beta > 2TF/\kappa_{\mathbb{H}}$ **holds** for virtually all channels of practical interest

Proof Technique

- $\mathrm{U}(W)$ and $\mathrm{L}(W)$ do **not** have the same first-order Taylor series expansion around $1/W = 0$

- We find a **new lower bound** with same asymptotic behavior as $\mathrm{U}(W)$

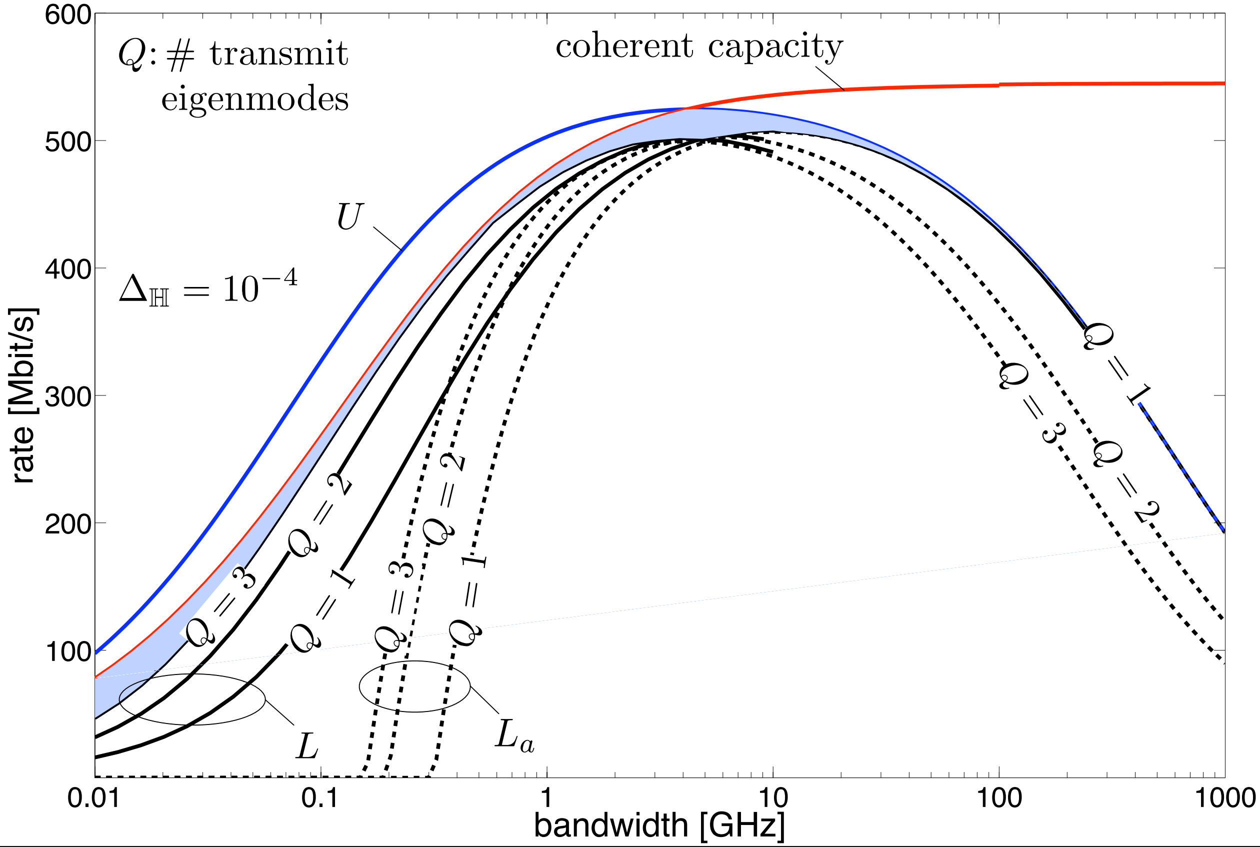
- To get this lower bound, we use signals with
  - **uniformly distributed** i.i.d. phase
  - **block-constant** magnitude randomly toggled on and off

  and apply a result from [*Prelov and Verdú, 2004*]

- Information is encoded in **both** phase and magnitude

The MIMO Setting: A Numerical Example for a 3×3 System

$\mathrm{U}(W)$ and $\mathrm{L}(W)$ for **spatially correlated MIMO** channels [*Schuster et al., 2008*]



Remarks on the MIMO Results

- Possible to identify the **optimal combination** of **bandwidth** and number of **transmit antennas**

- In the wideband regime:
  - It is **optimal** to use **only one** transmit antenna for spatially uncorrelated channels
  - For spatially correlated channels, rank-one statistical **beamforming along** the **strongest transmit eigenmode** is optimal
  - Both **transmit** and **receive** correlation are **beneficial** in the wideband regime

- Multiple antennas at the transmitter are **not beneficial** for **ultra-wideband** systems

Further Results & Open Problems

- The impact of the approximation error in the discretization ✓
- Capacity (bounds) for scattering functions that are not compactly supported ✓
- The high-SNR regime ✓
- The overspread case
- Peak constraints on the continuous-time transmit signal $x(t)$

Thank You

Sincerely yours

Publications

- G. Durisi, U. G. Schuster, H. Bölcskei, and S. Shamai (Shitz), "Noncoherent capacity of underspread fading channels," *IEEE Trans. Inf. Theory*, Apr. 2008, submitted [Online]. Available: `http://arxiv.org/abs/0804.1748`

- U. G. Schuster, G. Durisi, H. Bölcskei, and H. V. Poor, "Capacity bounds for peak-constrained multiantenna wideband channels," *IEEE Trans. Commun.*, Jan. 2008, submitted [Online]. Available: `http://arxiv.org/abs/0801.1002`