Time Delay Estimation on Parallel Fading Channels

Richard J. Kozick

Bucknell University

Collaboration: **Brian M. Sadler, Army Research Lab**

IEEE SAM 2008

Model: TDE on Parallel Channels

Complex signal received on N channels:

$$r_i(t) = \gamma_i \, s_i(t - \tau) \; + \; n_i(t), \;\; i = 1, \ldots, N.$$

Complex channel gains (flat) $\gamma = [\gamma_1, \ldots, \gamma_N]^T$

Time delay (common over N channels)

AWGN: complex, circular, PSD $\mathcal{N}_o$, iid over channels

<u>Parameters:</u>

Known signal shapes $s_i(t)$

Time delay, τ – unknown, deterministic

Channel gains, $\gamma = [\gamma_1, \ldots, \gamma_N]^T$: nuisance, various models, deterministic or random

Applications

$$r_i(t) = \gamma_i\, s_i(t-\tau) + n_i(t), \quad i = 1, \ldots, N.$$

- TDE with sum-of-tones (narrowband) signals
  - Multiple subbands, frequency selective (WB) channel
  - Subbands: flat, independent fading
  - Study tradeoffs with more bands (diversity)
- TDE with frequency-hopping signal
  - Process multiple hops
  - Study tradeoffs with more hops / shorter dwell

$$s_i(t) = \sum_{k=-K}^{K} a_{k,i}\, h\left[t - (i-1)LT_s - kT_s\right]$$

- Baseband model per channel (passband later)

Cramer-Rao Bounds (CRBs)

$$r_i(t) = \gamma_i \, s_i(t - \tau) + n_i(t), \quad i = 1, \ldots, N.$$

- Common form for various channel models:

$$\mathrm{CRB}(\hat{\tau}) = \frac{1}{2(2\pi B_s)^2 N \cdot \mathrm{SNR}_1}$$

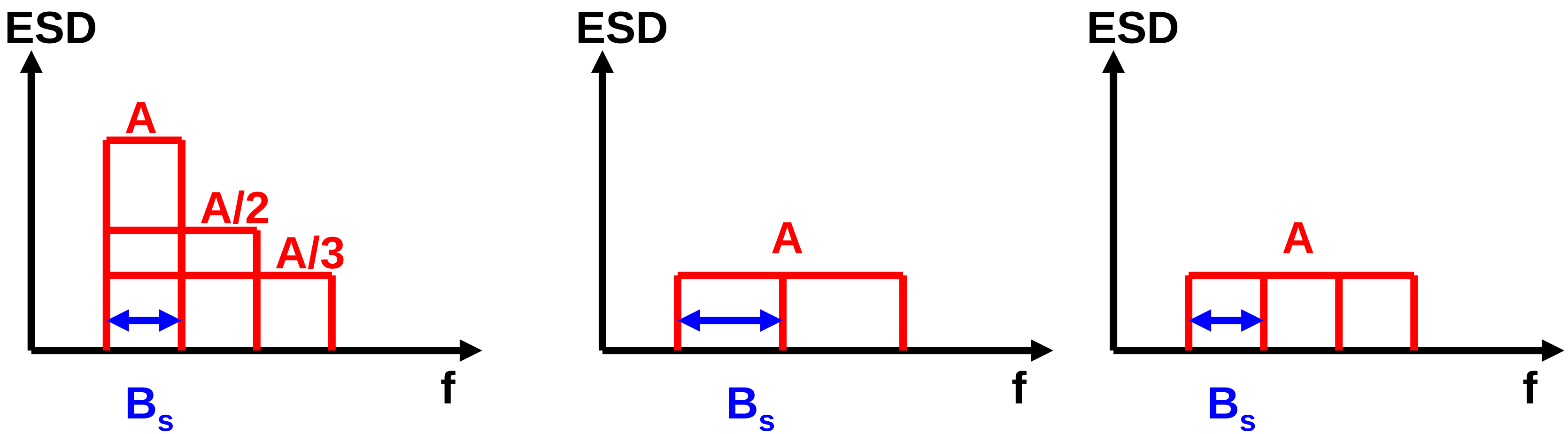
- SNR_1 = SNR per channel (average)
- B_s = signal bandwidth (curvature of correl. fun.)

- Depends on <u>total</u> signal energy over N channels, ($N^*\mathrm{SNR}_1$), even with random fading
- CRB indicates <u>no</u> diversity gain if ($N^*\mathrm{SNR}_1$) is fixed
- Tight bound?

More on the Model ...

$$r_i(t) = \gamma_i\, s_i(t-\tau) + n_i(t), \quad i = 1, \ldots, N.$$

$$\mathrm{CRB}(\hat{\tau}) = \frac{1}{2(2\pi B_s)^2 N \cdot \mathrm{SNR}_1}$$



- Add subbands with equal bandwidth (same signal shape)
- Dividing into subbands:
  less accuracy and violates flat fading per channel
- B_s = coherence BW of fading

Outline

- Analyze five channel model cases:
  - Maximum likelihood estimator (MLE)
  - Bounds: Cramer-Rao (CRB) & Ziv-Zakai (ZZB)
  - Mean-squared error (MSE) by simulation
- Study TDE accuracy vs. fading & diversity
- ZZB/MSE: Diversity ➔ lower SNR thresh.
- Other models:
  - TDE and array processing (bearing est.)
  - Passband vs. baseband signals

Model

- Complex signal for N channels:

$$r_i(t) = \gamma_i\, s_i(t - \tau) + n_i(t), \quad i = 1, \ldots, N.$$

Complex channel gains (flat) $\gamma = [\gamma_1, \ldots, \gamma_N]^T$

Time delay (common over N channels)

AWGN: complex, circular, PSD $\mathcal{N}_o$, iid over channels

<u>Parameters:</u>

Known signal shapes $s_i(t)$

Time delay, τ – unknown, deterministic

Channel gains, $\gamma = [\gamma_1, \ldots, \gamma_N]^T$: 5 model cases

Channel Model Cases

1. γ is known
2. γ is unknown, deterministic ("conditional")
3. Rayleigh fading, iid: ("unconditional")

$$\gamma \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$$

Rician is 2 & 3

1A. Case 1 averaged over $\gamma \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$
➔ Fading w/ perfect knowledge of γ

2A. Case 2 averaged over $\gamma \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$
➔ Fading w/ imperfect estimate of γ

- Signal correlation functions: $$\rho_i(\xi) = \int s_i(t-\xi)^* \, s_i(t)\, dt,$$
- Signal energy & mean-square bandwidth:

$$\rho(0) = \int \left|s_i(t)\right|^2 \, dt, \qquad B_s^2 = \int \frac{f^2 \, \left|S_i(f)\right|^2}{\rho(0)} \, df$$

- Average SNR per channel:
  - deterministic average (cases 1 & 2)
  - mean over fading (cases 3, 1A, 2A)

$$\mathrm{SNR}_{\mathrm{det}}\left(\boldsymbol{\gamma}\right) = \left(\rho(0)/N\right) \sum_{i=1}^{N} \frac{|\gamma_i|^2}{\mathcal{N}_o}$$

$$\overline{\mathrm{SNR}} = \frac{\rho(0)\, E\left\{|\gamma_i|^2\right\}}{\mathcal{N}_o} = \frac{\rho(0)\, \sigma_s^2}{\mathcal{N}_o} = E_{\boldsymbol{\gamma}} \left\{\mathrm{SNR}_{\mathrm{det}}\left(\boldsymbol{\gamma}\right)\right\}$$

Assumptions

A1 The signal autocorrelation functions in (2) are the same on each channel, $\rho_i(\xi) \triangleq \rho(\xi)$, $i = 1, \ldots, N$, so the signal energy is

$$\rho_i(0) = \int |s_i(t)|^2 \, dt \triangleq \rho(0), \quad i = 1, \ldots, N. \tag{9}$$

A2 The signals have the property $\int s_i(t)\, \dot{s}_i(t)^* \, dt = 0$, where $\dot{s}_i(t) = \frac{d}{dt} s_i(t)$.

A3 The mean-squared signal bandwidth is the same on each channel and is denoted by B_s Hz, with definition

$$B_s^2 = \int \frac{|\dot{s}_i(t)|^2}{\rho(0)\ (2\pi)^2} \, dt = \int \frac{f^2\ |S_i(f)|^2}{\rho(0)} \, df, \tag{10}$$

where $S_i(f)$ is the Fourier transform of $s_i(t)$.

Log-likelihood Functions

$$Y_i^{\mathrm{COH}}(\xi) = \mathrm{Re} \int \gamma_i^* \, s_i(t-\xi)^* \, r_i(t) \, dt$$

$$Y_i^{\mathrm{NONCOH}}(\xi) = \left| \int s_i(t-\xi)^* \, r_i(t) \, dt \right| .$$

$$\mathcal{L}_c \left(\boldsymbol{R} \,|\, \gamma; \tau \right) = \frac{1}{\mathcal{N}_o} \sum_{i=1}^{N} \left[2 \, Y_i^{\mathrm{COH}}(\tau) - |\gamma_i|^2 \rho(0) \right]$$

$$\mathcal{L}_u \left(\boldsymbol{R} \,|\, \tau \right) = \frac{\sigma_s^2}{\mathcal{N}_o \left(\mathcal{N}_o + \sigma_s^2 \, \rho(0) \right)} \sum_{i=1}^{N} Y_i^{\mathrm{NONCOH}}(\tau)^2$$

Maximum Likelihood Estimators

- MLEs are coherent or noncoherent matched filters (MF):

$$Z_{\mathrm{C}}(\xi) = \sum_{i=1}^{N} \mathrm{Re}\left(\int_{-\infty}^{\infty} \gamma_i^* \, s_i(t-\xi)^* \, r_i(t) \, dt \right)$$

$$Z_{\mathrm{N}}(\xi) = \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} s_i(t-\xi)^* \, r_i(t) \, dt \right|^2$$

Summary of Results

	Channel Model				
	1 (KN.)	**2** (UNK.)	**3** (RAN.)	**1A**	**2A**
MLE	Coh MF	NonCoh MF	NonCoh MF	Coh MF	NonCoh MF
CRB	CRB_1	$\mathrm{CRB}_2 = \mathrm{CRB}_1$	CRB_3	MCRB	ACRB = MCRB
ZZB	ZZB_1	ZZB_2	ZZB_3	ZZB_{1A}	$\mathrm{ZZB}_{2A} = \mathrm{ZZB}_3$
MSE	MSE_1	MSE_2	MSE_3	MSE_{1A}	$\mathrm{MSE}_{2A} = \mathrm{MSE}_3$

- MLE: Coherent MF for cases 1 & 1A
- Modified & asymptotic CRB (MCRB & ACRB) for random nuisance parameters
- Four distinct ZZB & MSE cases

Cramer-Rao Bounds (CRB)

$$\mathrm{CRB}_1\left(\widehat{\tau}\,|\,\boldsymbol{\gamma}\right) = \mathrm{CRB}_2 = \frac{1}{2\,\left(2\pi\,B_s\right)^2\;N\,\mathrm{SNR}_{\mathrm{det}}\left(\boldsymbol{\gamma}\right)}$$

Random channel: Little impact on CRB!

$$\mathrm{CRB}_3\left(\widehat{\tau}\right) = \frac{1+\left(\overline{\mathrm{SNR}}\right)^{-1}}{2\,\left(2\pi\,B_s\right)^2\;N\,\overline{\mathrm{SNR}}}$$

Depend on <u>total</u> signal energy over N channels

$$\mathrm{CRB}_{1A}\left(\widehat{\tau}\right) \approx \mathrm{MCRB}\left(\widehat{\tau}\right) \triangleq \frac{1}{2\,\left(2\pi\,B_s\right)^2\;N\,\overline{\mathrm{SNR}}}$$

$$\mathrm{CRB}_{2A}\left(\widehat{\tau}\right) \approx \mathrm{ACRB}\left(\widehat{\tau}\right) \triangleq \frac{1}{2\,\left(2\pi\,B_s\right)^2\;N\,\overline{\mathrm{SNR}}}.$$

CRBs with Fixed Total SNR

- Fix total SNR over the channels:

$$\overline{\mathrm{SNR}}_{\mathrm{TOT}} = N \cdot \overline{\mathrm{SNR}}, \quad \overline{\mathrm{SNR}} = \overline{\mathrm{SNR}}_{\mathrm{TOT}} / N$$

- CRB for Rayleigh fading (case 3):

$$\mathrm{CRB}_3(\hat{\tau}) = \begin{cases} \dfrac{1}{2(2\pi B_s)^2 \, \overline{\mathrm{SNR}}_{\mathrm{TOT}}}, & N \ll \overline{\mathrm{SNR}}_{\mathrm{TOT}} \\ \dfrac{N}{2(2\pi B_s)^2 \, \overline{\mathrm{SNR}}^2_{\mathrm{TOT}}}, & N \gg \overline{\mathrm{SNR}}_{\mathrm{TOT}} \end{cases}$$

  – No diversity advantage for small N

  – Penalty if N is too large (spread fixed energy over many channels)

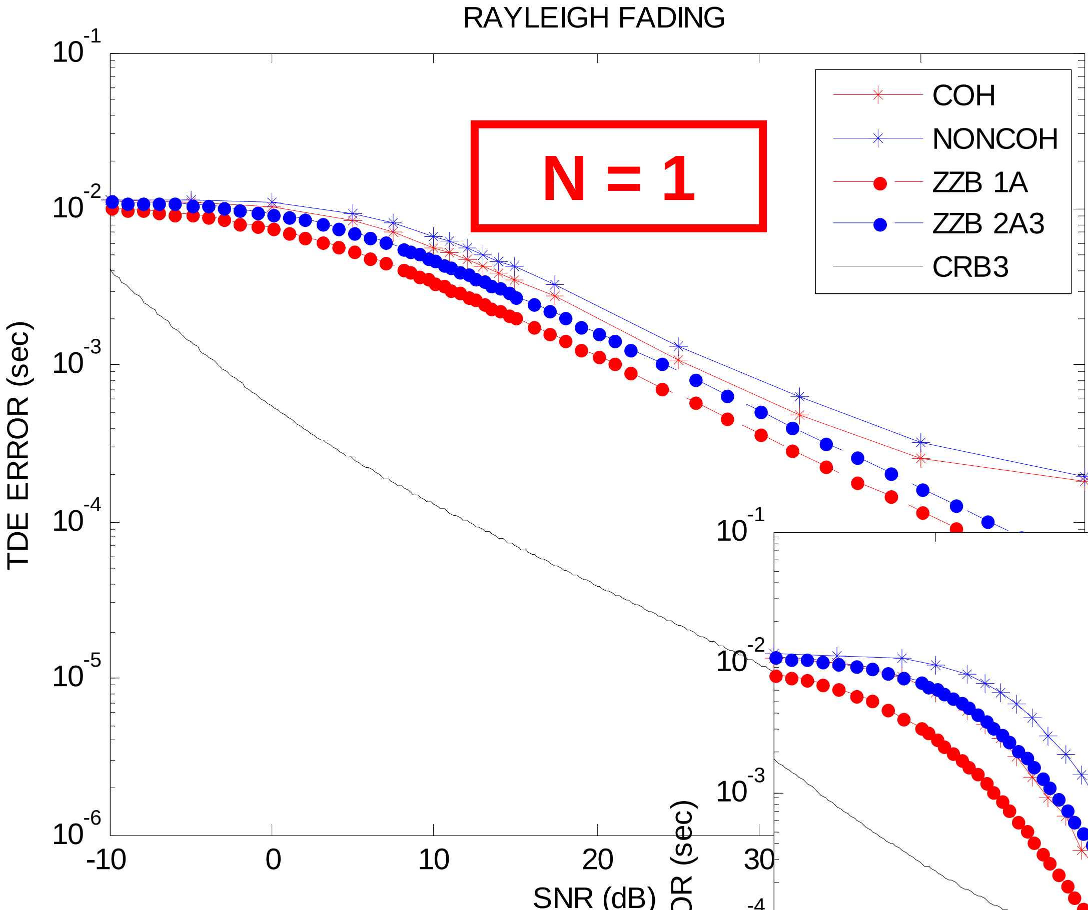
Miller-Chang Bound (MCB)

$$\mathrm{CRB}_1\left(\widehat{\tau}\,|\,\boldsymbol{\gamma}\right) = \mathrm{CRB}_2 = \frac{1}{2\ \left(2\pi\, B_s\right)^2\ N\,\mathrm{SNR}_{\mathrm{det}}\left(\boldsymbol{\gamma}\right)}$$

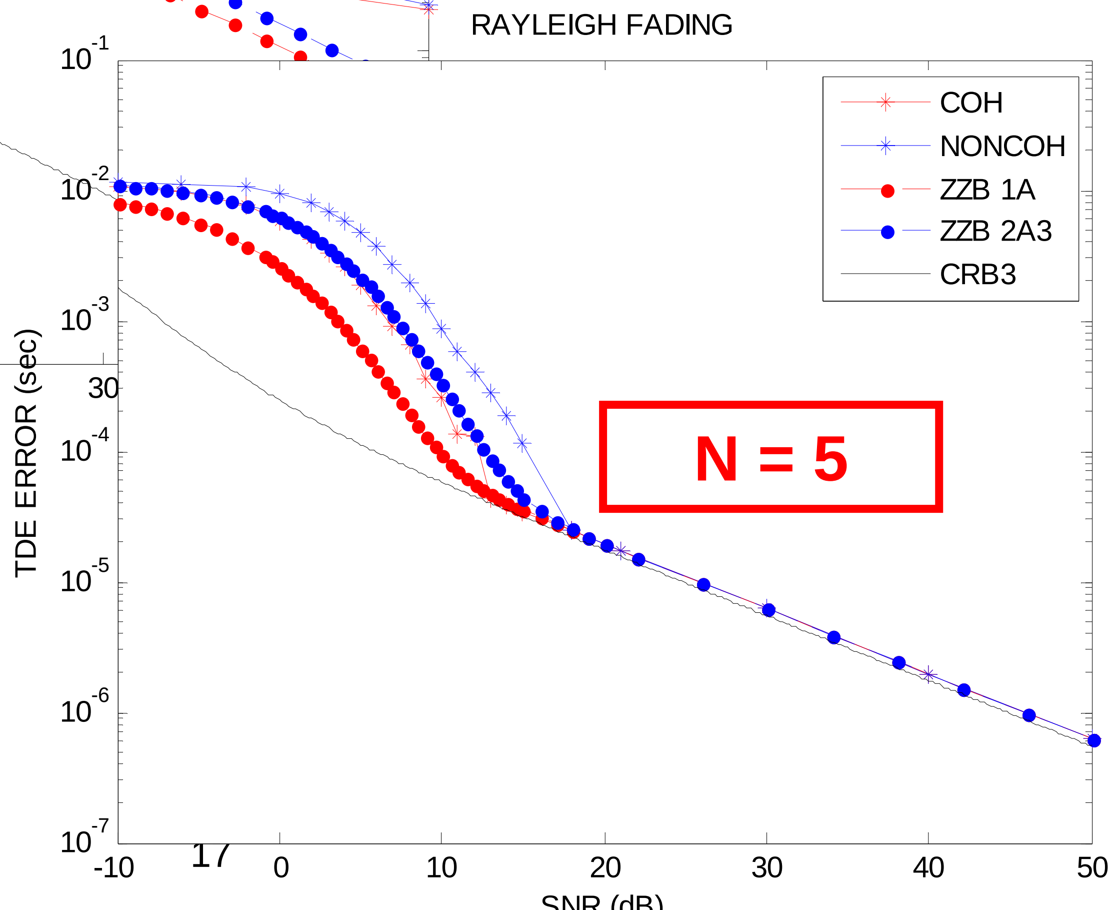
- MCB for estimators that are unbiased for <u>every</u> realization of γ (locally unbiased):

$$\mathrm{MCB}\left(\hat{\tau}\right) = E_{\gamma}\left\{\mathrm{CRB}_1\left(\hat{\tau}|\boldsymbol{\gamma}\right)\right\} = \frac{1}{2\left(2\pi B_s\right)^2\left(N-1\right)\overline{\mathrm{S}}\overline{\mathrm{NR}}}$$

Diverges for N=1



Need tighter bound: low SNR & diversity ➔ Ziv-Zikai bound



Ziv-Zakai Bound (ZZB)

- Tighter than CRB: low SNR and threshold
- Accounts for ambiguities (sidelobes)
- Developed by Ziv, Zakai, Chazan, Bellini, Tartara (1969-1974)
- Applied to time-delay estimation by Weiss and Weinstein (1983, …)
- Excellent reference on this and other bounds:
  - H.L. Van Trees & K.L. Bell, *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/ Tracking*, Wiley, 2007.

Ziv-Zakai Bound (ZZB)

- Hypothesis test:

$$H_0: \quad r_i(t) = \gamma_i \, s_i(t-a) \, + \, n_i(t), \quad i = 1, \ldots, N$$
$$H_1: \quad r_i(t) = \gamma_i \, s_i \, (t - (a+\theta)) \, + \, n_i(t).$$

- P_e = minimum probability of error
- ZZB is a lower bound on the MSE:

$$\text{ZZB}\,(\widehat{\tau}) \geq \frac{1}{D} \int_0^D \theta \; \mathcal{V}[(D-\theta) P_e(\theta)] \; d\theta$$

Ziv-Zakai Bound (ZZB)

$$\mathrm{ZZB}\left(\widehat{\tau}\right) \geq \frac{1}{D}\int_0^D \theta \; \mathcal{V}[(D-\theta)P_e(\theta)] \; d\theta$$

- ZZB is a Bayesian bound
- Assume *a priori* pdf on τ is uniform with length D
- At high SNR, ZZB converges to standard CRB for deterministic τ
- Need P_e expressions for each case
  ➔ Available from BER analysis w/ fading
- Numerical integration usually req'd for ZZB

$$P_{e,1}\left(\theta \,|\, \boldsymbol{\gamma}\right) = Q\left(\sqrt{N \cdot \mathrm{SNR}_{\mathrm{det}}\left(\boldsymbol{\gamma}\right) \cdot \left(1 - \frac{\mathrm{Re}\left[\rho(\theta)\right]}{\rho(0)}\right)}\right)$$

$$P_{e,2}\left(\theta\right) = S\left(a(\theta, \boldsymbol{\gamma}, N), b(\theta, \boldsymbol{\gamma}, N), N\right)$$

$$P_{e,1A}\left(\theta\right) = R\left(\mu(\theta), N\right)$$

$$P_{e,3}\left(\theta\right) = P_{e,2A}\left(\theta\right) = R\left(\nu(\theta), N\right).$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-u^2/2)\, du$$

$$R(x, N) = \left(\frac{1-x}{2}\right)^{2N-1} \sum_{i=0}^{N-1} \binom{2N-1}{i} \left(\frac{1+x}{1-x}\right)^{i}$$

Coherent MF

$$\mu(\theta) = \left[1 + \frac{2}{\overline{\mathrm{SNR}} \cdot \left(1 - \mathrm{Re}\left[\rho(\theta)\right] / \rho(0)\right)}\right]^{-1/2}$$

Noncoherent MF

$$\nu(\theta) = \left[1 + \frac{2\left[1 + \left(\overline{\mathrm{SNR}}\right)^{-1}\right]}{\left(\overline{\mathrm{SNR}}/2\right) \cdot \left(1 - \left|\rho(\theta)/\rho(0)\right|^{2}\right)}\right]^{-1/2}$$

Noncoherent MF, no fading (case 2):

$$S(\alpha, \beta, N) = \frac{1}{2} + \frac{1}{2^{2N-1}} \sum_{i=1}^{N} \binom{2N-1}{N-i} \left[Q_i(\alpha, \beta) - Q_i(\beta, \alpha) \right]$$

$$Q_i(\alpha, \beta) = \text{ generalized } i^{th} \text{ order Marcum Q-function [12]}$$

$$a(\theta, \boldsymbol{\gamma}, N) = \left[\frac{N}{2} \cdot \mathrm{SNR}_{\mathrm{det}}(\boldsymbol{\gamma}) \cdot \left(1 - \sqrt{1 - |\rho(\theta)/\rho(0)|^2} \right) \right]^{1/2}$$

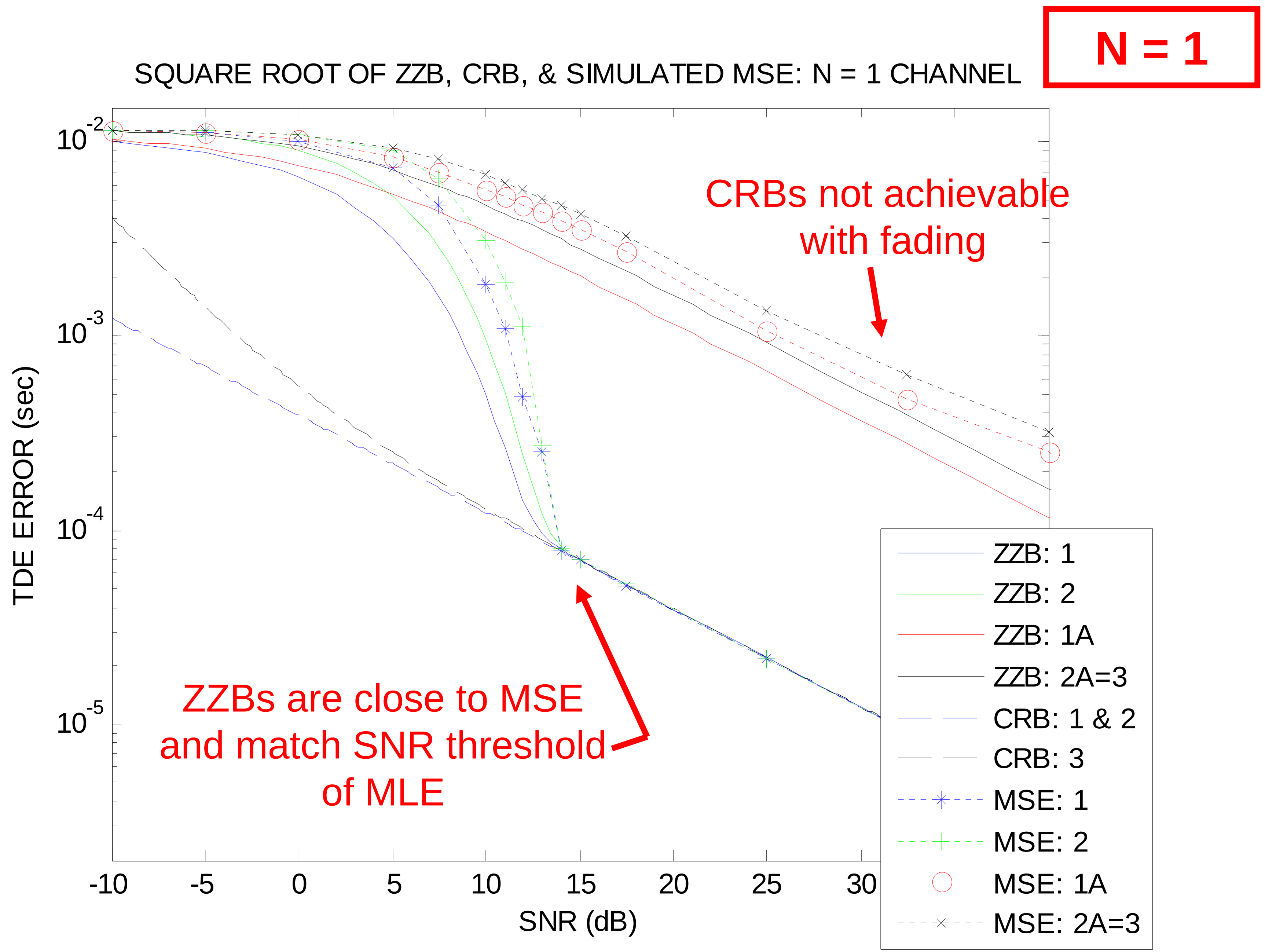
$$b(\theta, \boldsymbol{\gamma}, N) = \left[\frac{N}{2} \cdot \mathrm{SNR}_{\mathrm{det}}(\boldsymbol{\gamma}) \cdot \left(1 + \sqrt{1 - |\rho(\theta)/\rho(0)|^2} \right) \right]^{1/2}$$

P_e for (N*SNR) Fixed

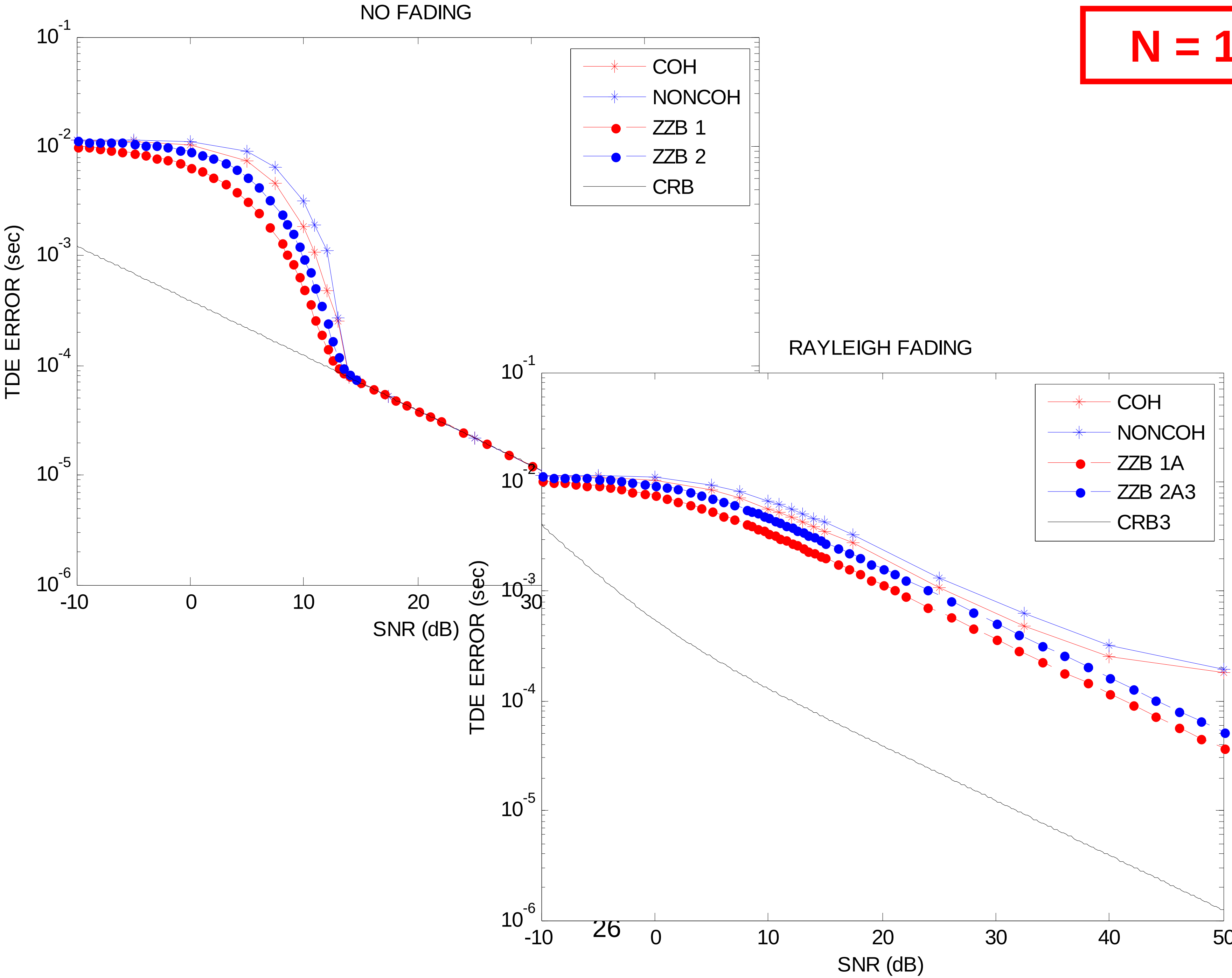
- Case 1 (known channel) depends on (N*SNR) ➔ no loss or gain from N>1
- Case 2 (unknown det. channel) is max. for N = 1 ➔ loss due to noncoherent combining
- Dependence on N with fading is nontrivial
- Cases 1A (known channel) and 2A=3: Roughly 3 dB difference in SNR

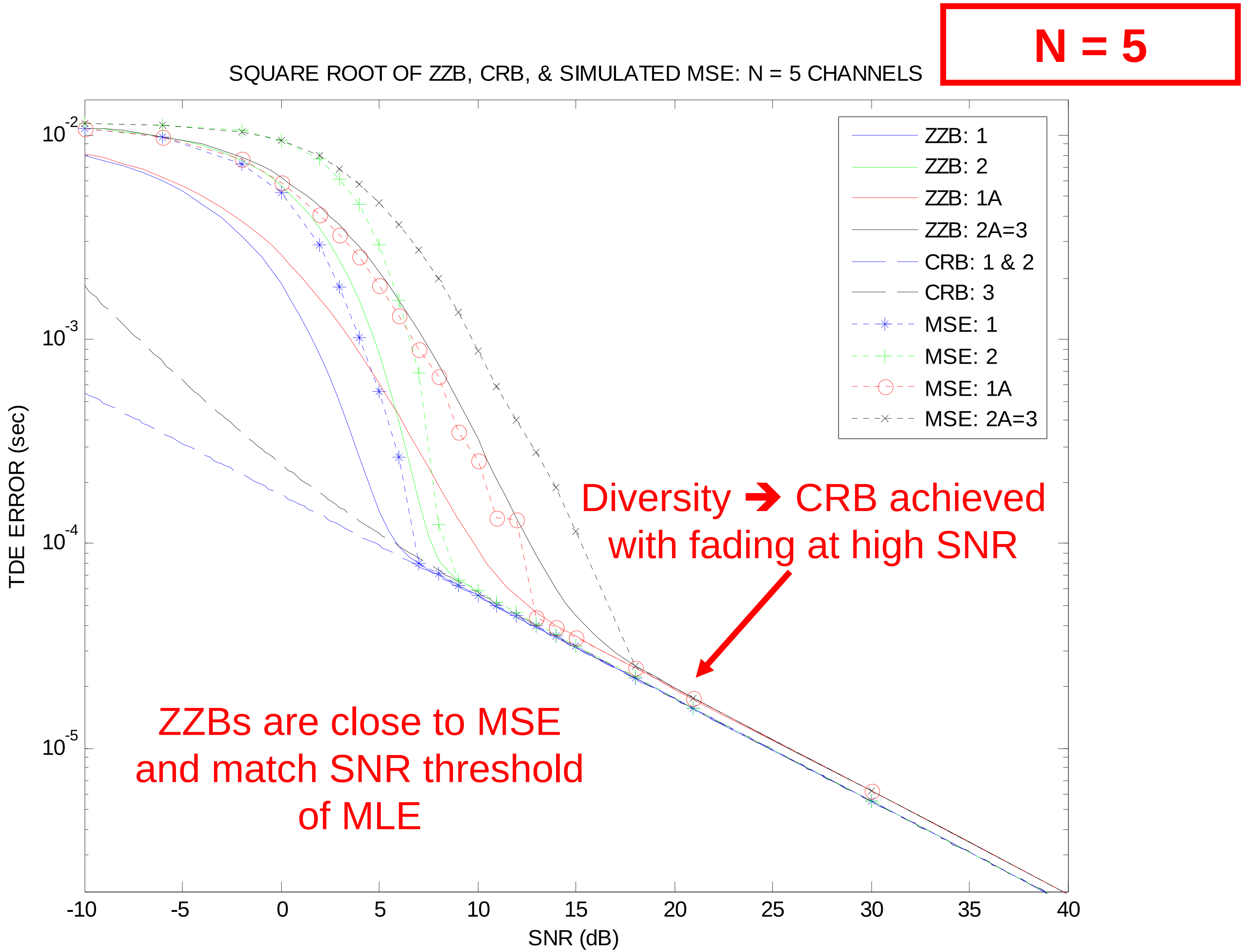
Simulations

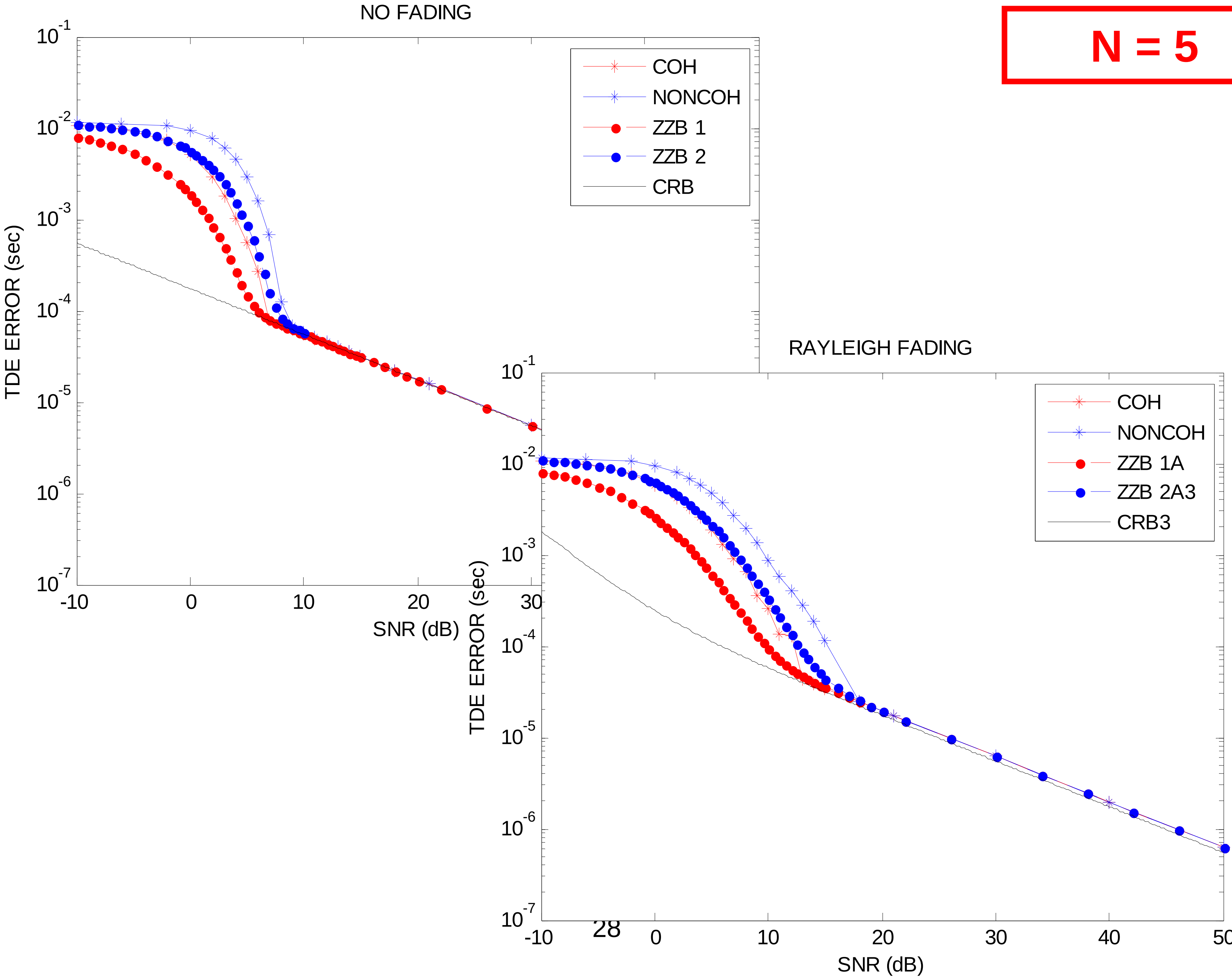
- Signal: square-root raised-cosine, zero excess BW, period 10^{-3} s
- SNR per channel is constant with N
- MSE: 10,000 to 22,000 Monte Carlo runs
- Channels: N=1, N=5, and N=3


N = 1
SQUARE ROOT OF ZZB, CRB, & SIMULATED MSE: N = 1 CHANNEL
TDE ERROR (sec)
10^-2
10^-3
10^-4
10^-5
-10
-5
0
5
10
15
20
25
30
SNR (dB)
CRBs not achievable with fading
ZZBs are close to MSE and match SNR threshold of MLE
ZZB: 1
ZZB: 2
ZZB: 1A
ZZB: 2A=3
CRB: 1 & 2
CRB: 3
MSE: 1
MSE: 2
MSE: 1A
MSE: 2A=3

N = 1

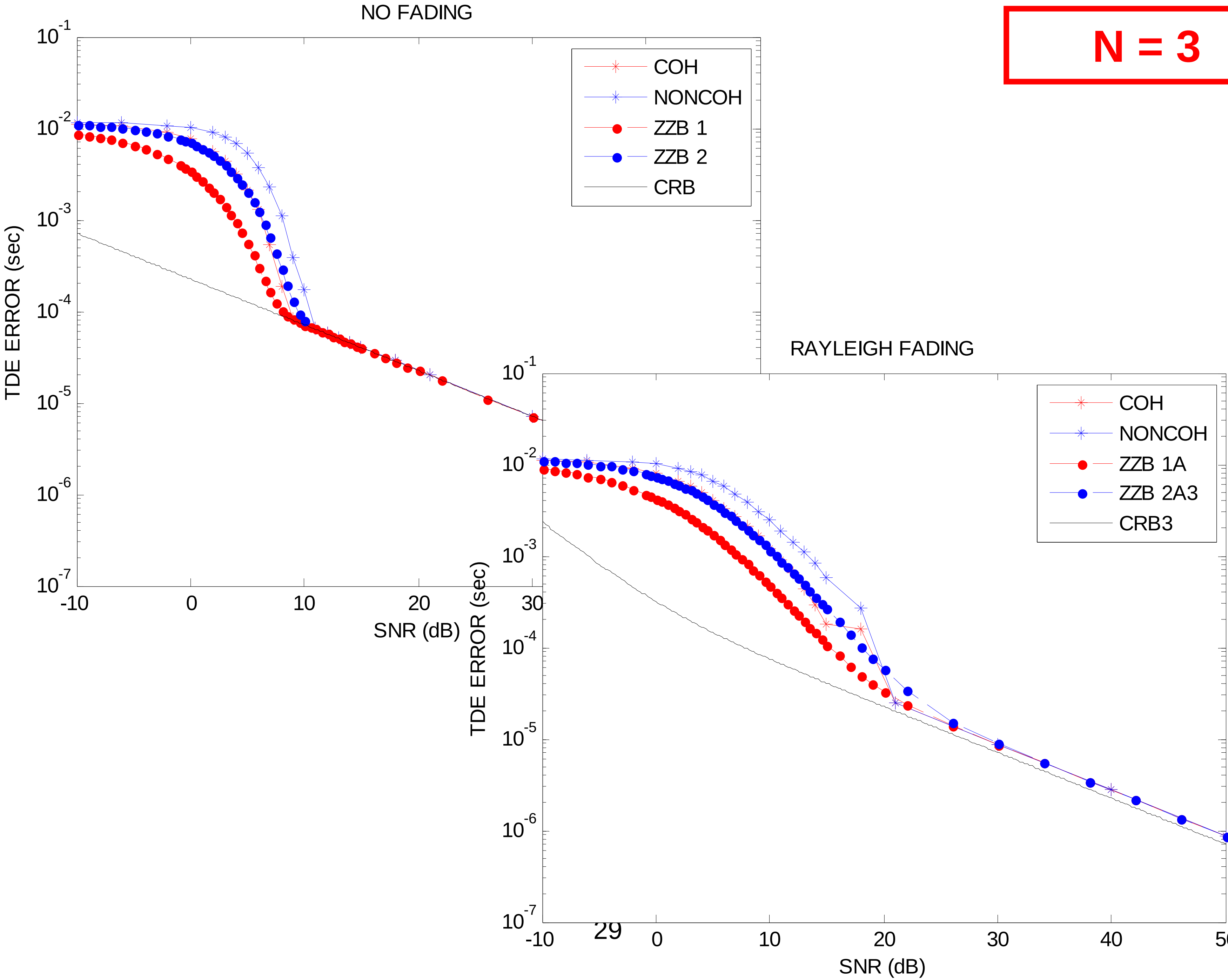

NO FADING
TDE ERROR (sec)
SNR (dB)
COH
NONCOH
ZZB 1
ZZB 2
CRB
RAYLEIGH FADING
TDE ERROR (sec)
SNR (dB)
COH
NONCOH
ZZB 1A
ZZB 2A3
CRB3


N = 5
SQUARE ROOT OF ZZB, CRB, & SIMULATED MSE: N = 5 CHANNELS
ZZB: 1
ZZB: 2
ZZB: 1A
ZZB: 2A=3
CRB: 1 & 2
CRB: 3
MSE: 1
MSE: 2
MSE: 1A
MSE: 2A=3
Diversity → CRB achieved with fading at high SNR
ZZBs are close to MSE and match SNR threshold of MLE
TDE ERROR (sec)
SNR (dB)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
-10
-5
0
5
10
15
20
25
30
35
40


NO FADING
TDE ERROR (sec)
SNR (dB)
COH
NONCOH
ZZB 1
ZZB 2
CRB
RAYLEIGH FADING
TDE ERROR (sec)
SNR (dB)
COH
NONCOH
ZZB 1A
ZZB 2A3
CRB3

N = 5

N = 3



NO FADING
TDE ERROR (sec)
SNR (dB)
COH
NONCOH
ZZB 1
ZZB 2
CRB
RAYLEIGH FADING
TDE ERROR (sec)
SNR (dB)
COH
NONCOH
ZZB 1A
ZZB 2A3
CRB3

TDE and Array Processing

$$\text{TDE}: \quad r_i(t) = s(t-\tau)\gamma_i + n_i(t), \quad i = 1, \ldots, N$$

$$\text{Array}: \; \mathbf{r}_i = \mathbf{s}(\tau)\gamma_i + \mathbf{n}_i, \quad i = 1, \ldots, N$$

$$\text{or} \quad \mathbf{r}_i = \mathbf{a}(\theta)s_i + \mathbf{n}_i$$

- Apply array processing results for conditional and unconditional models (CRBs, MLEs)
- ZZB and threshold analysis:
  - K. Bell, Y. Ephraim and H. L. Van Trees, “Explicit Ziv-Zakai lower bounds for bearing estimation,” *IEEE Trans. SP*, Nov. 1996.
  - F. Athley, “Threshold Region Performance of Maximum Likelihood DOA Estimation for a Single Source,” *IEEE Trans. SP*, Apr. 2005.

P_e Bound for Rayleigh Fading

- Exact:

$$P_{e,3}(\theta)=\left(\frac{1-\nu(\theta)}{2}\right)^{2N-1}\sum_{i=0}^{N-1}\binom{2N-1}{i}\left(\frac{1+\nu(\theta)}{1-\nu(\theta)}\right)^{i}$$

$$\nu(\theta)=\left[1+\frac{1}{(\alpha/4)\left(1-\left|\rho(\theta)/\rho(0)\right|^{2}\right)}\right]^{-1/2},\quad \alpha=\frac{\overline{\mathrm{S}}\overline{\mathrm{NR}}}{1+\frac{1}{\overline{\mathrm{S}}\overline{\mathrm{NR}}}}$$

- Tight lower bound:

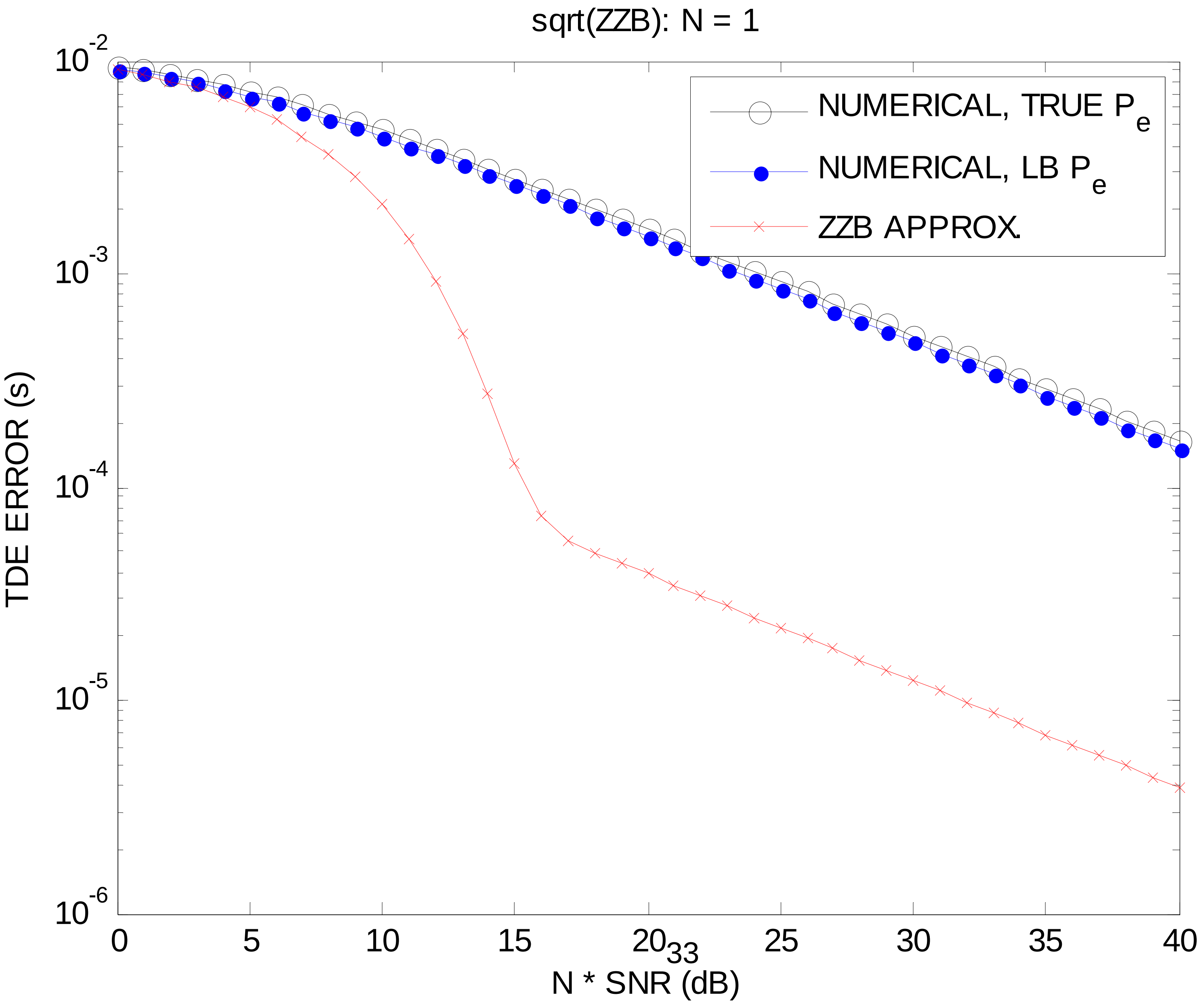
$$P_{e,3}(\theta)\geq\left\{\left[1-\nu(\theta)^{2}\right]\exp\left[\nu(\theta)^{2}\right]\right\}^{N}Q\left(\nu(\theta)\sqrt{2N}\right)$$

ZZB Approximations

$$\mathrm{ZZB}\left(\widehat{\tau}\right) \geq \frac{1}{D}\int_0^D \theta\ \mathcal{V}[(D-\theta)P_e(\theta)]\ d\theta$$

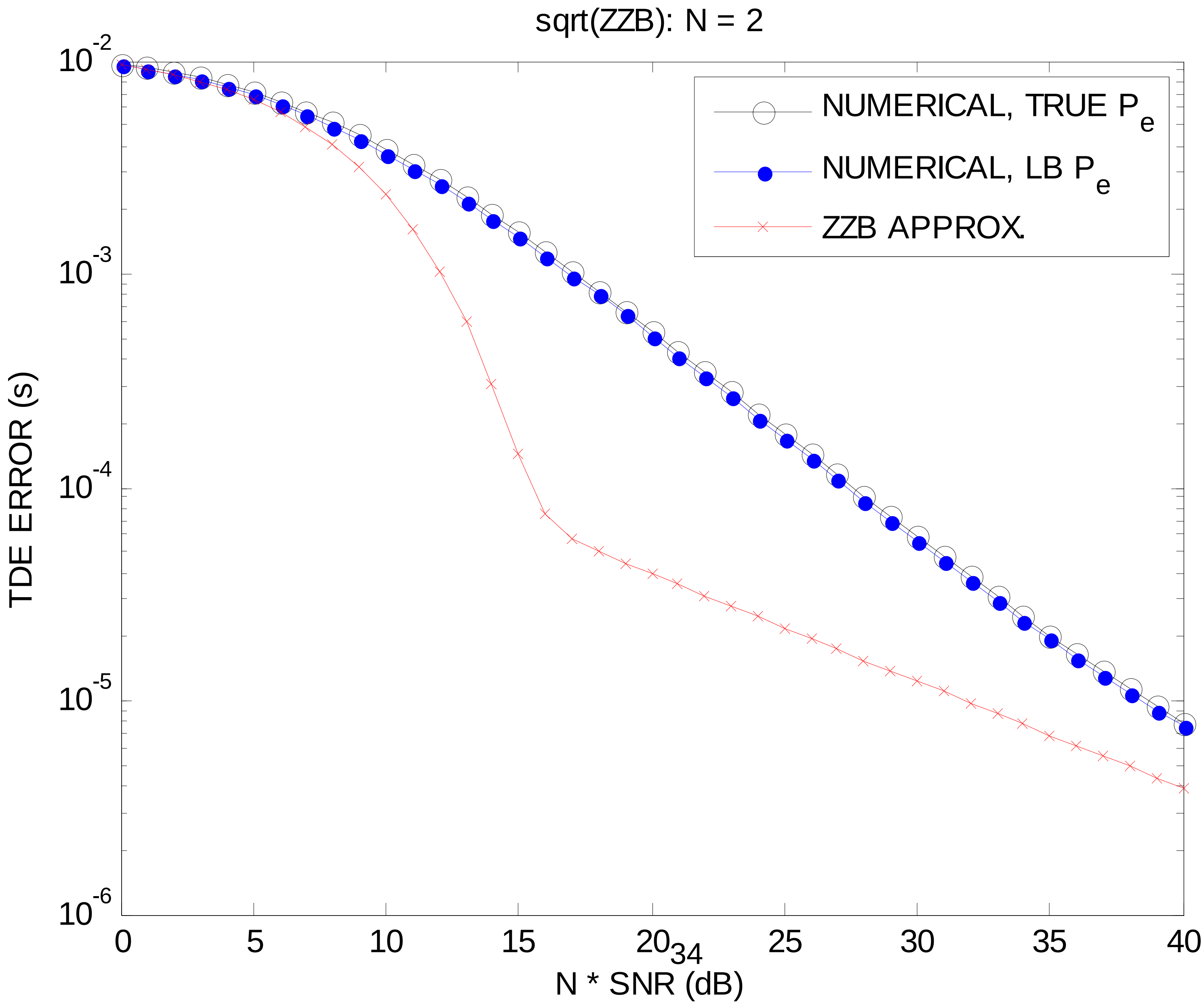
- Numerical integration with
  - Exact P_e
  - Lower bound on P_e
- Closed-form approximation from
  - K. Bell, Y. Ephraim and H. L. Van Trees, "Explicit Ziv-Zakai lower bounds for bearing estimation," *IEEE Trans. SP*, Nov. 1996.
  - Tight for large N in TDE problem, not small N

N = 1



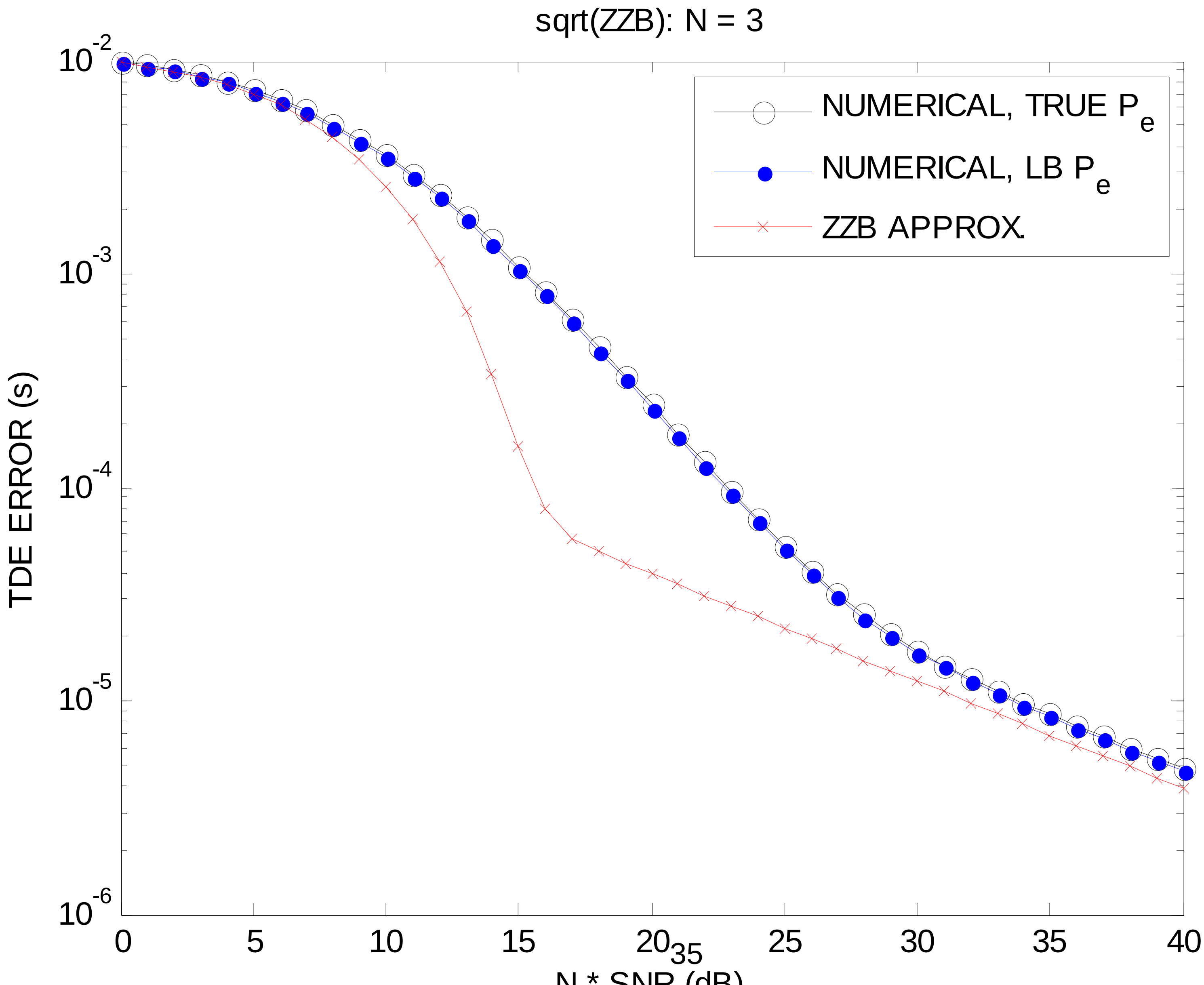
sqrt(ZZB): N = 1
TDE ERROR (s)
N * SNR (dB)
NUMERICAL, TRUE P_e
NUMERICAL, LB P_e
ZZB APPROX.
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30
35
40

N = 2



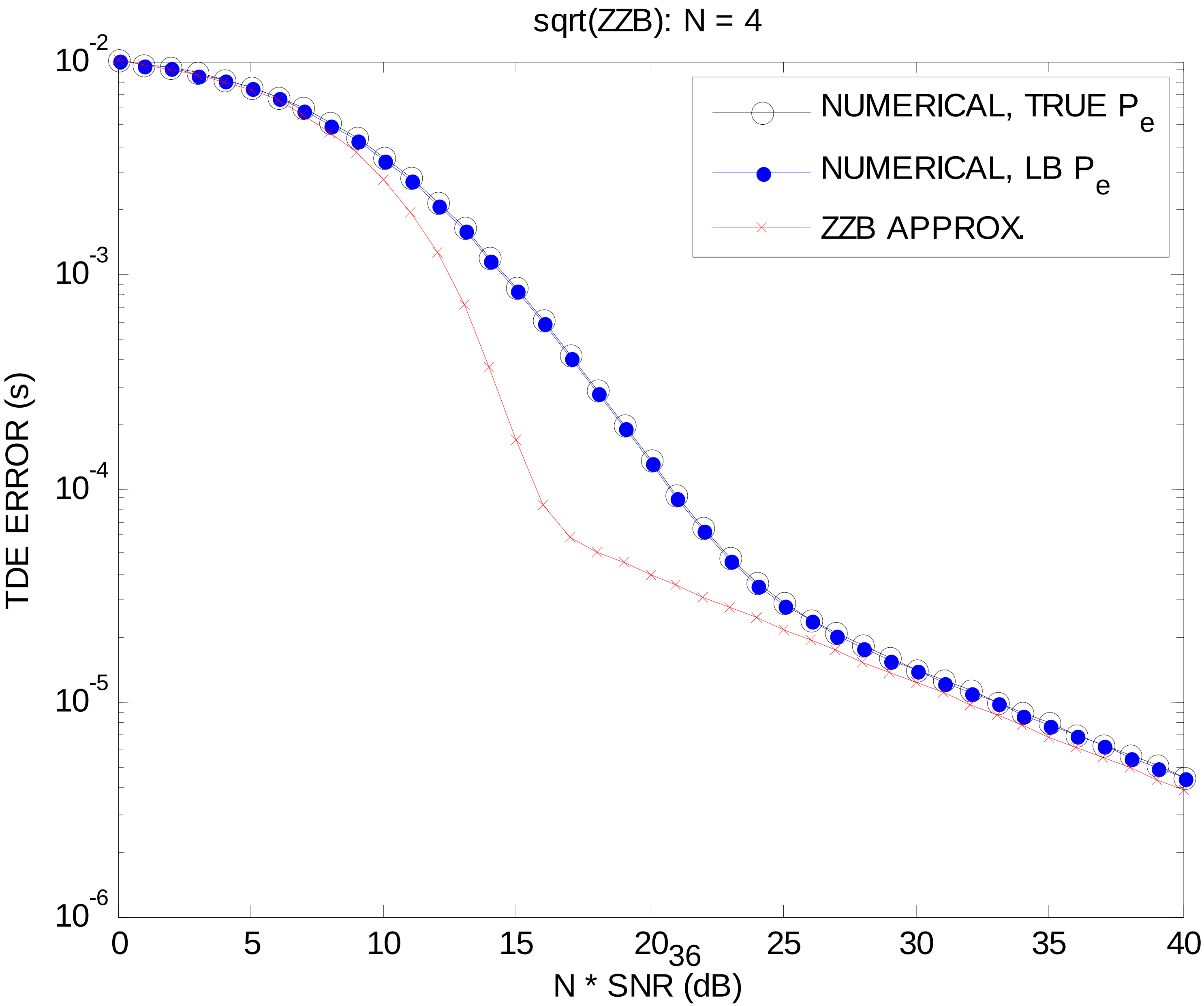
sqrt(ZZB): N = 2
NUMERICAL, TRUE P_e
NUMERICAL, LB P_e
ZZB APPROX.
TDE ERROR (s)
N * SNR (dB)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30
35
40

N = 3

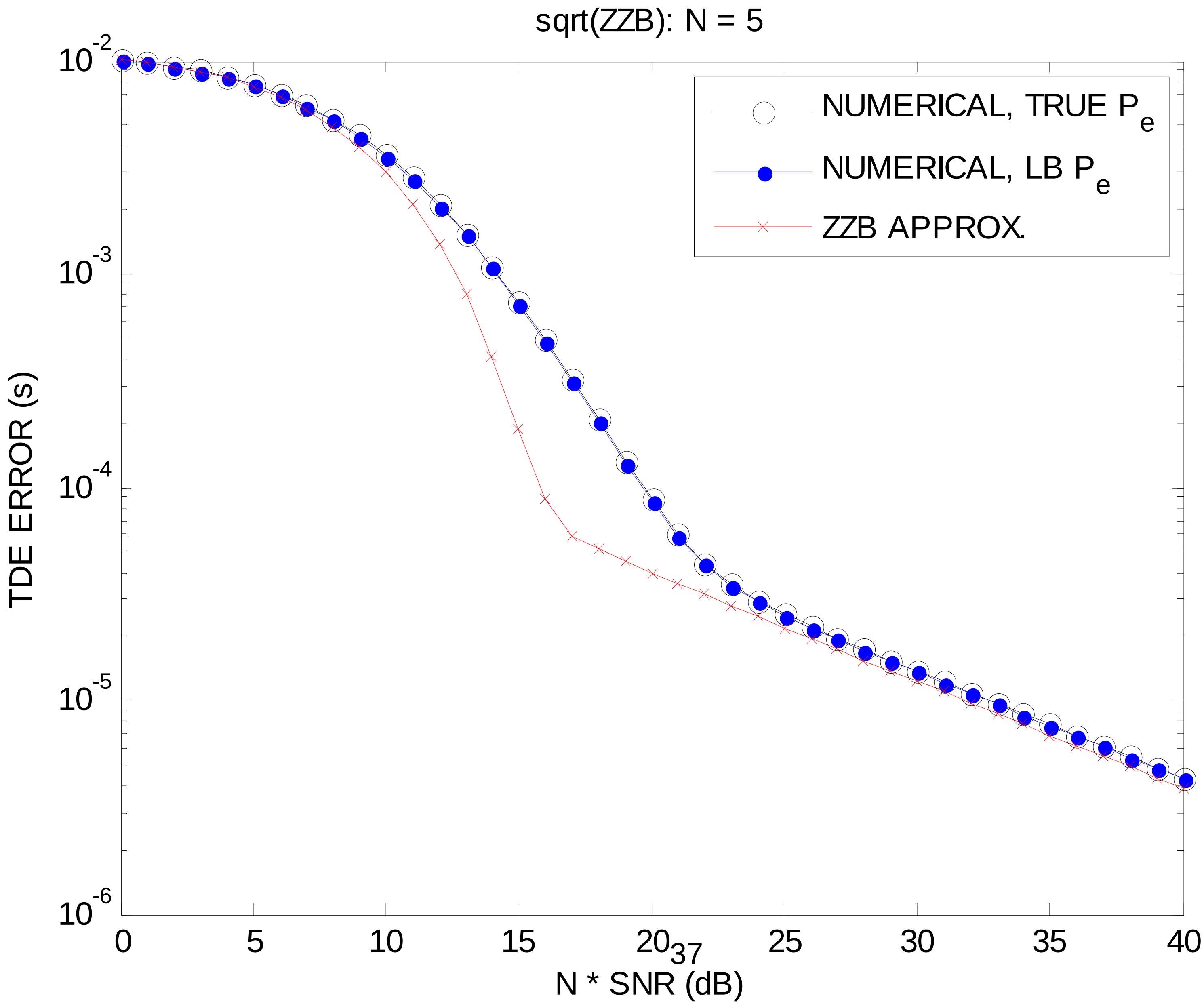


sqrt(ZZB): N = 3
TDE ERROR (s)
N * SNR (dB)
NUMERICAL, TRUE P_e
NUMERICAL, LB P_e
ZZB APPROX.
10^-2
10^-3
10^-4
10^-5
10^-6
0
5
10
15
20
25
30
35
40

N = 4

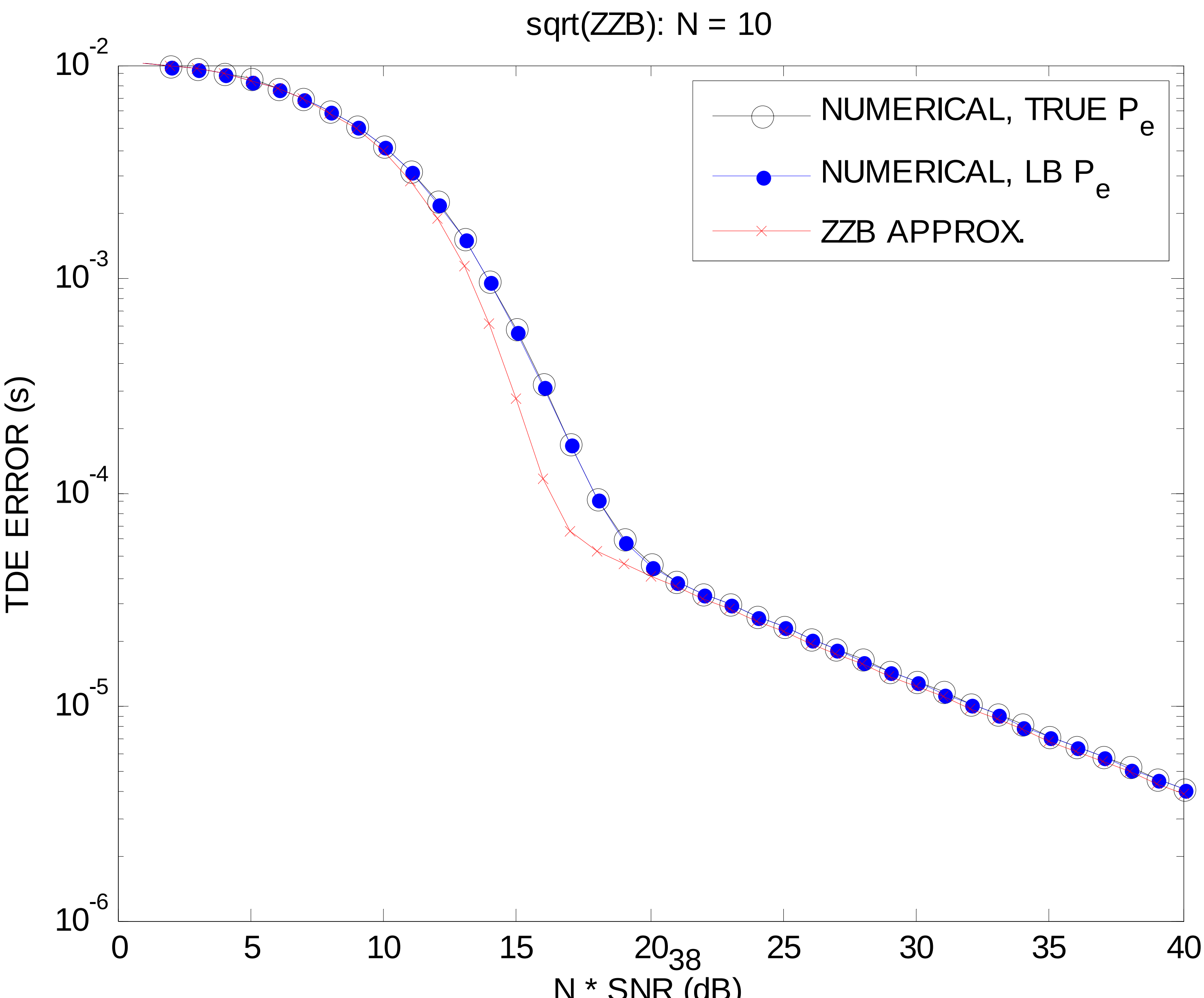

sqrt(ZZB): N = 4
NUMERICAL, TRUE P_e
NUMERICAL, LB P_e
ZZB APPROX.
TDE ERROR (s)
N * SNR (dB)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30
35
40

N = 5

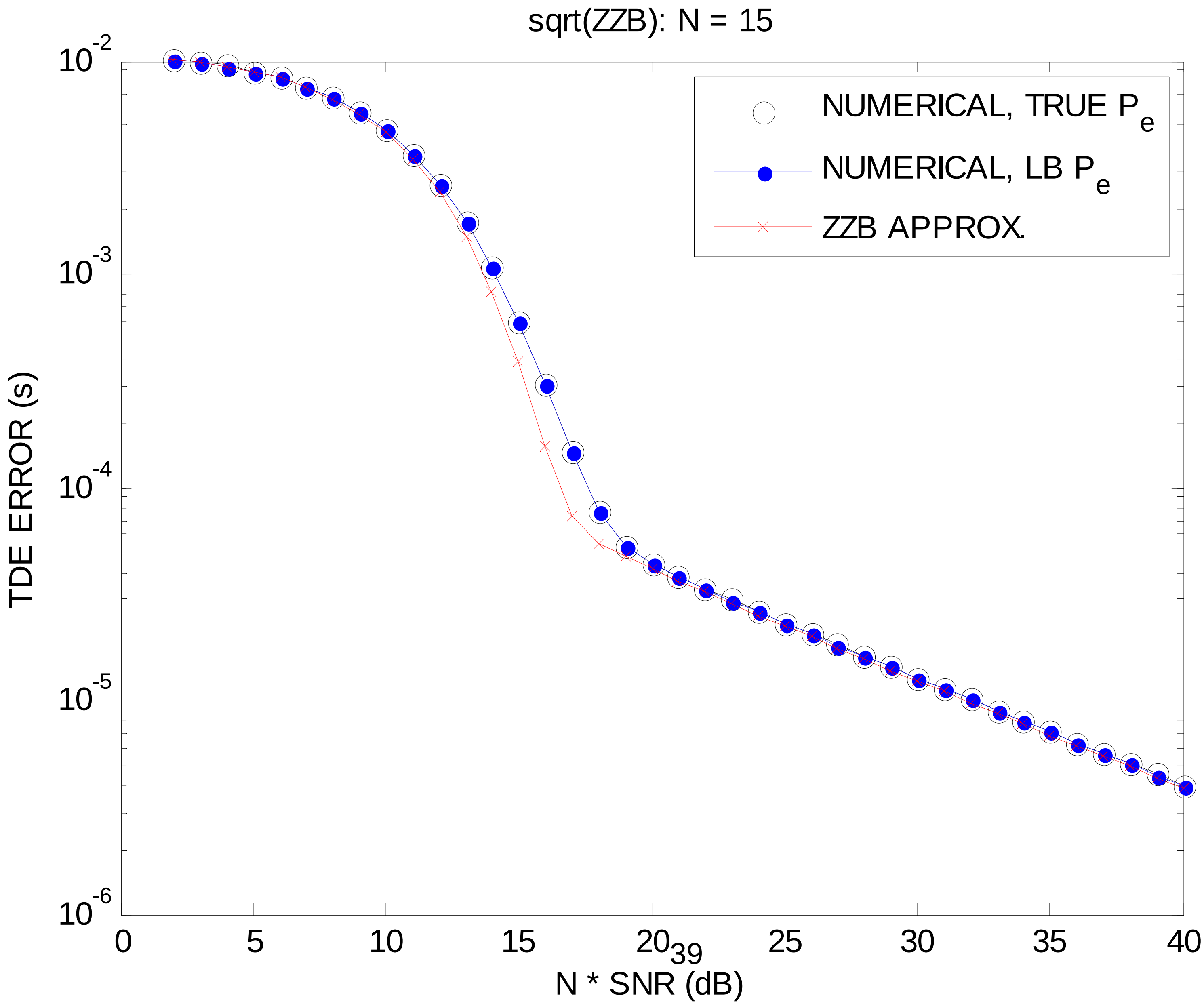


sqrt(ZZB): N = 5
NUMERICAL, TRUE P_e
NUMERICAL, LB P_e
ZZB APPROX.
TDE ERROR (s)
N * SNR (dB)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30
35
40

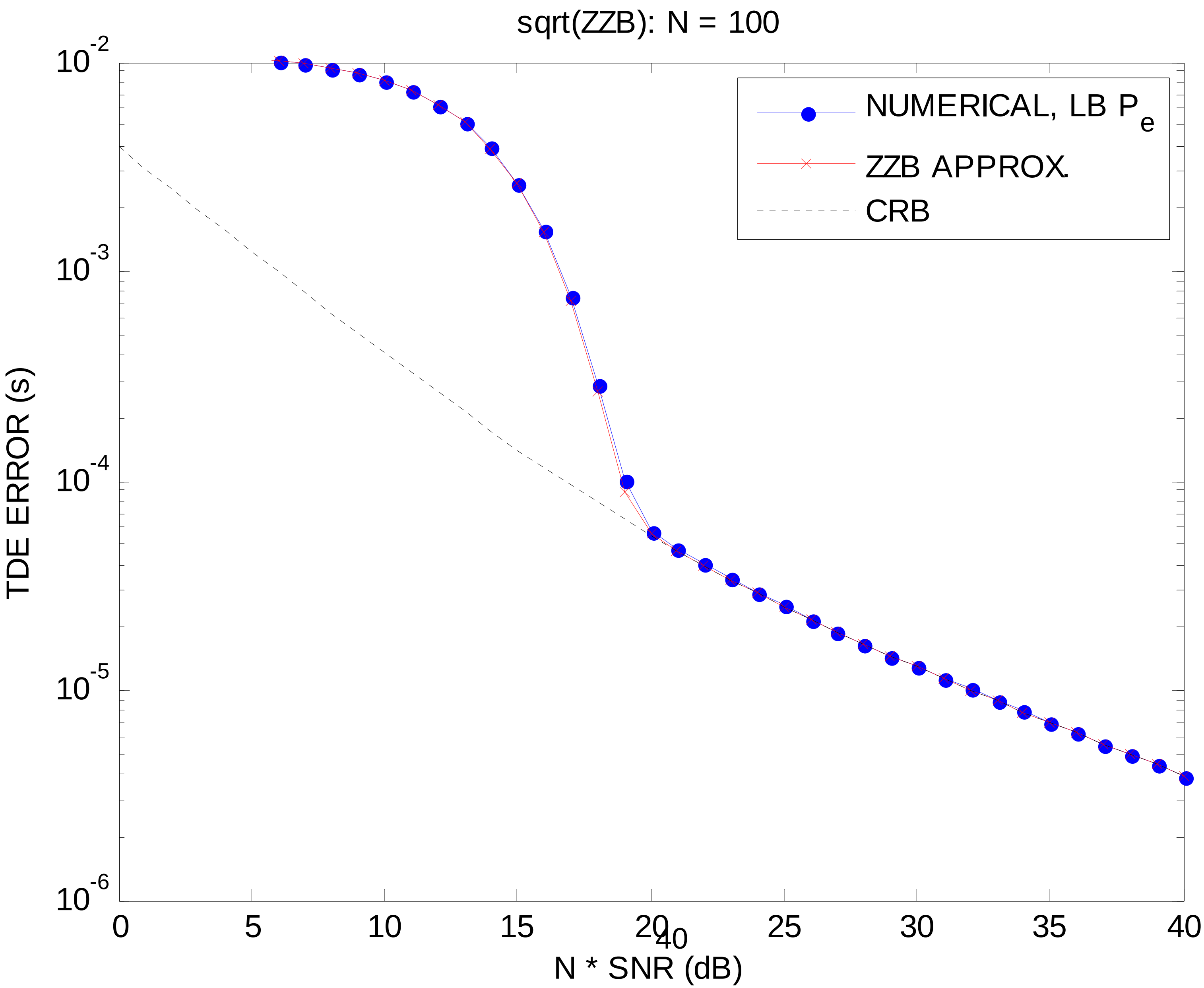
N = 10


sqrt(ZZB): N = 10
NUMERICAL, TRUE P_e
NUMERICAL, LB P_e
ZZB APPROX.
TDE ERROR (s)
10^-2
10^-3
10^-4
10^-5
10^-6
0
5
10
15
20
25
30
35
40
N * SNR (dB)

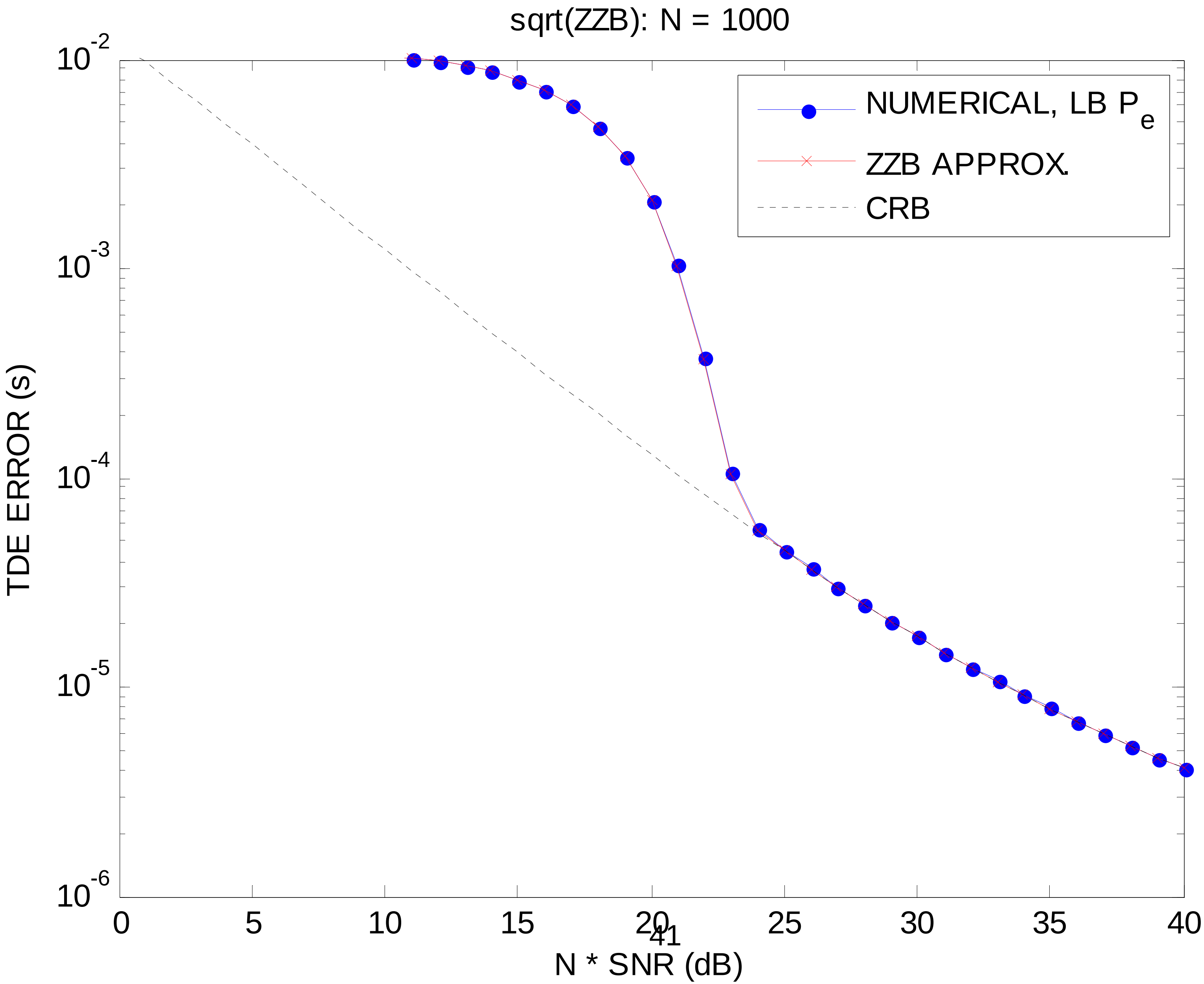
N = 15


sqrt(ZZB): N = 15
TDE ERROR (s)
N * SNR (dB)
NUMERICAL, TRUE Pe
NUMERICAL, LB Pe
ZZB APPROX.
10-2
10-3
10-4
10-5
10-6
0
5
10
15
20
25
30
35
40

N = 100

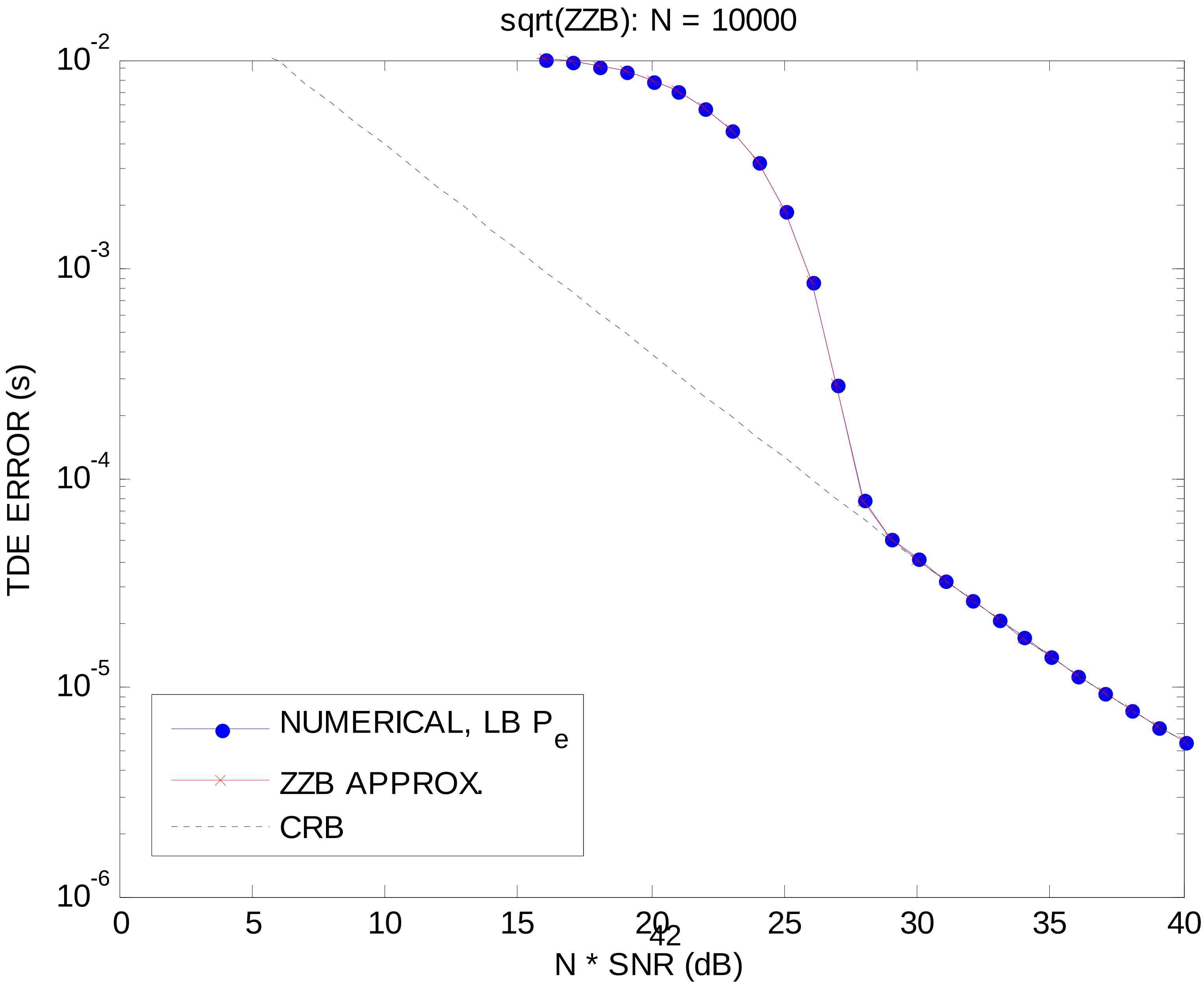

sqrt(ZZB): N = 100
TDE ERROR (s)
N * SNR (dB)
NUMERICAL, LB Pe
ZZB APPROX.
CRB
10-2
10-3
10-4
10-5
10-6
0
5
10
15
20
25
30
35
40

N = 1,000



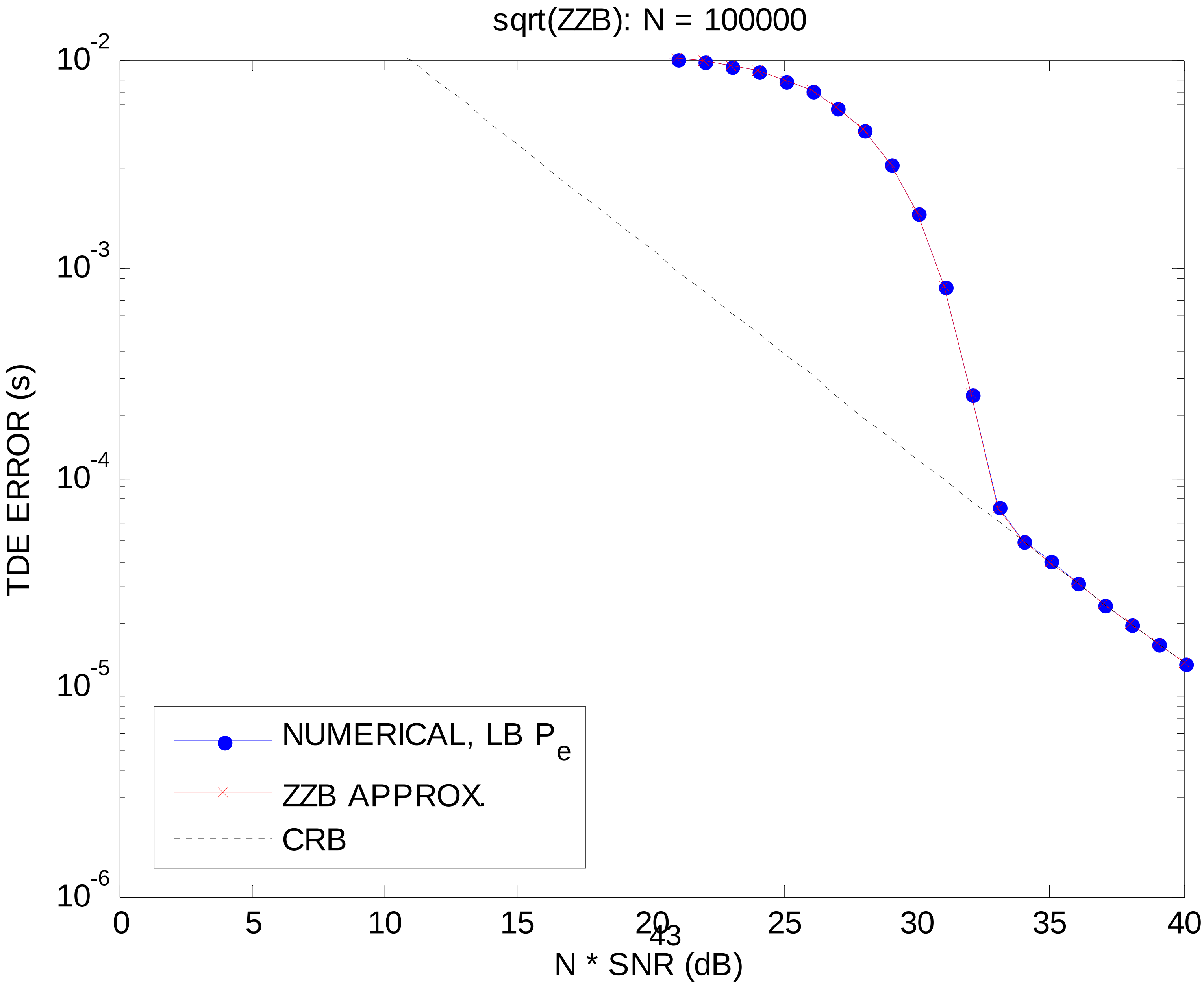
sqrt(ZZB): N = 1000
NUMERICAL, LB Pe
ZZB APPROX.
CRB
TDE ERROR (s)
N * SNR (dB)
10-2
10-3
10-4
10-5
10-6
0
5
10
15
20
25
30
35
40

N = 10,000

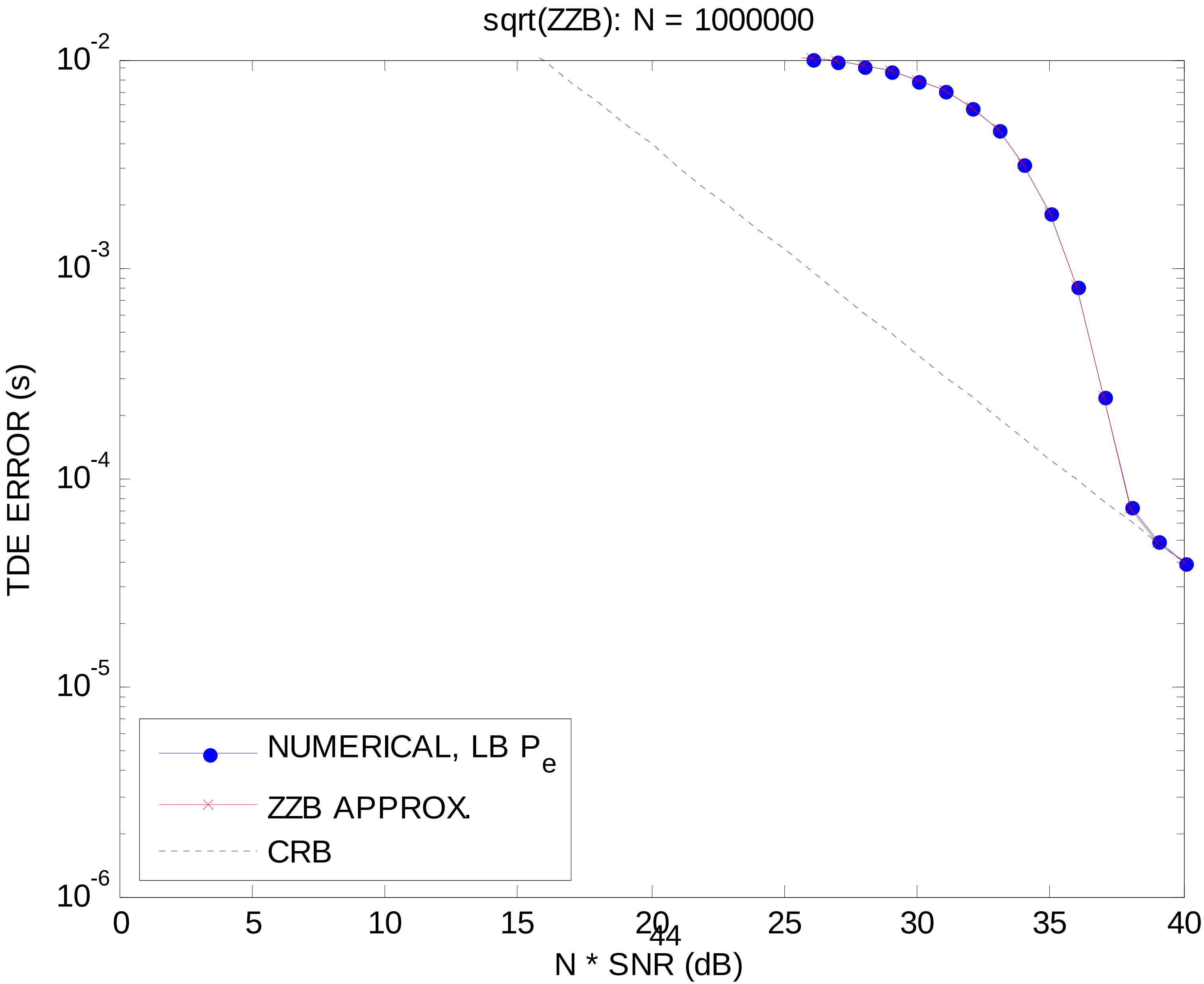


sqrt(ZZB): N = 10000
TDE ERROR (s)
N * SNR (dB)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
5
10
15
20
25
30
35
40
NUMERICAL, LB P_e
ZZB APPROX.
CRB

N = 100,000


sqrt(ZZB): N = 100000
TDE ERROR (s)
10^-2
10^-3
10^-4
10^-5
10^-6
0
5
10
15
20
25
30
35
40
N * SNR (dB)
NUMERICAL, LB P_e
ZZB APPROX.
CRB

N = 1e6



sqrt(ZZB): N = 1000000
TDE ERROR (s)
10^-2
10^-3
10^-4
10^-5
10^-6
0
5
10
15
20
25
30
35
40
N * SNR (dB)
NUMERICAL, LB P_e
ZZB APPROX.
CRB

ZZB for Fixed Total SNR

- Threshold SNR improves for N=1, 2, …
  then degrades for large N
  ➔ Optimum N? Diminishing returns?
- Analytical characterization?
- ZZB quantifies diversity gain for TDE over parallel, fading channels

Baseband / Passband Models

- Passband affects only cases 1 and 1A (known channel)
  - CRB includes carrier frequencies
  - ZZB similar to baseband if signals are narrowband
- For other cases (unknown channel): Passband MLE, CRB, and ZZB are identical to baseband

Open Problems

- Further explore applying array processing results (CRBs, ZZBs, MLE performance, etc.) to this problem
- Analytical characterization of threshold for TDE vs. SNR and N ➔ Optimum N
- Connect with TDE analysis for frequency selective, wideband fading channels (Xu & Sadler)