The mathematician Ramanujan, and digital signal processing

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1887 - 1920

1877 - 1947

Self-educated mathematician from India

Grew up in poverty in Tamil Nadu

His genius was discovered by G. H. Hardy

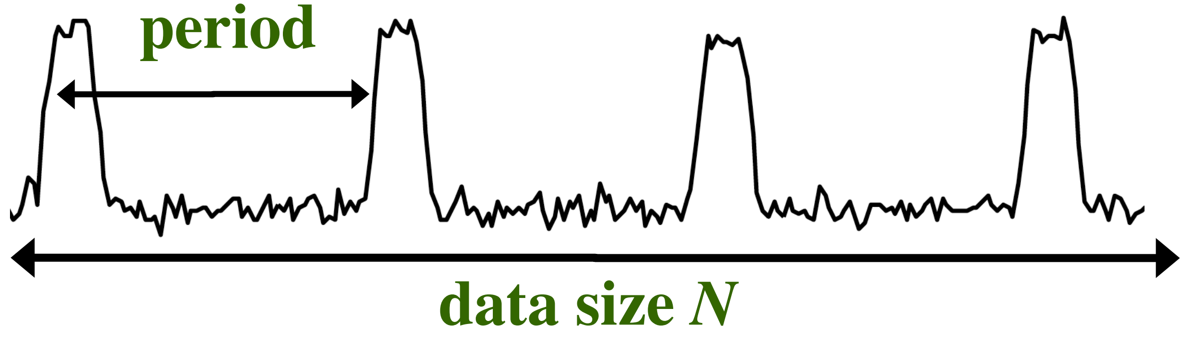
Worked with Hardy in 1914 - 1919

Outline

- Ramanujan sums (RS): 1918
- Representing periodic signals
- New integer bases, integer projections
- Ramanujan periodic dictionaries
- Bounds on period estimation
- Conclusions

Motivation

Consider a periodic signal



DFT representation: $$x(n) = \sum_{k=0}^{N-1} X[k] e^{j(2\pi k/N)n}$$

period N
or a divisor of N

Example: $N = 32$; $q = 1, 2, 4, 8, 16, 32$

Very few periods represented

Ramanujan-sum representation: $$x(n) = \sum_{q=1}^{N} a_q c_q(n)$$

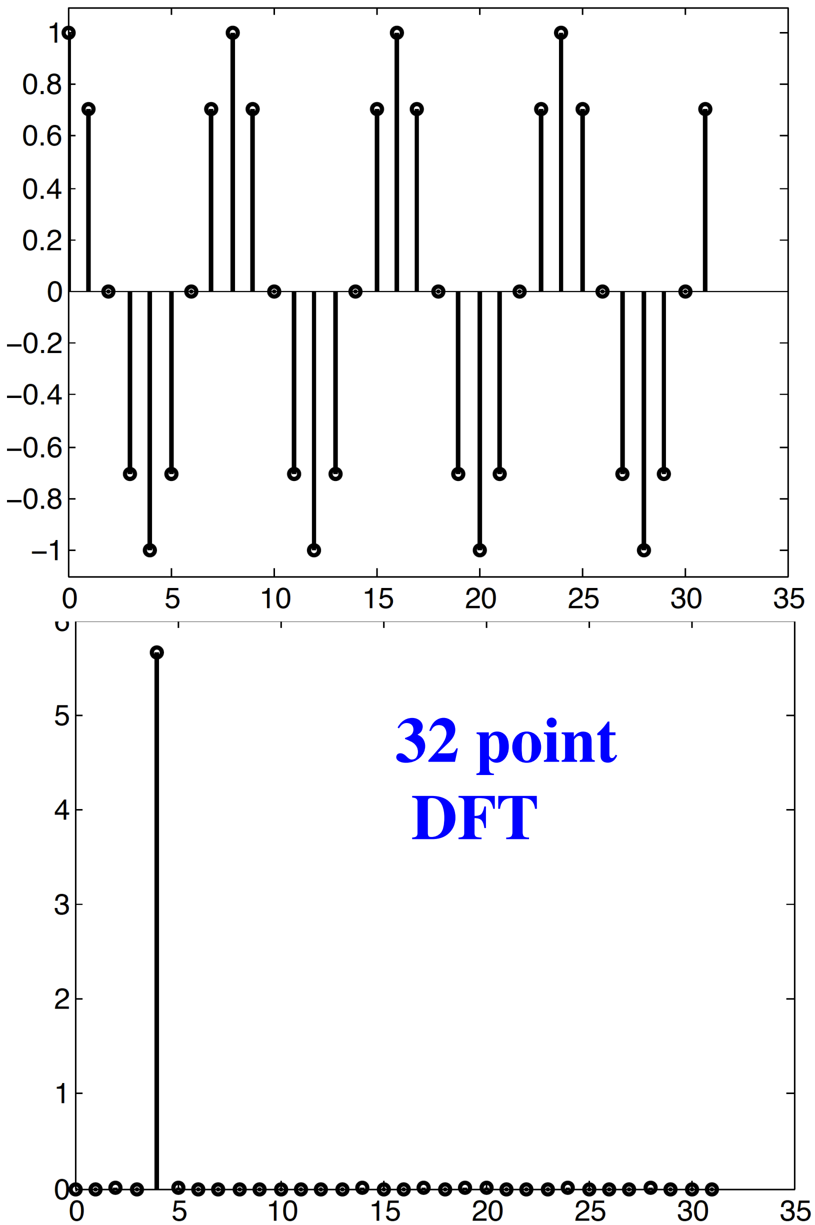
period q

Example: $N = 32$; $q = 1, 2, 3, 4, 5, \ldots 32$

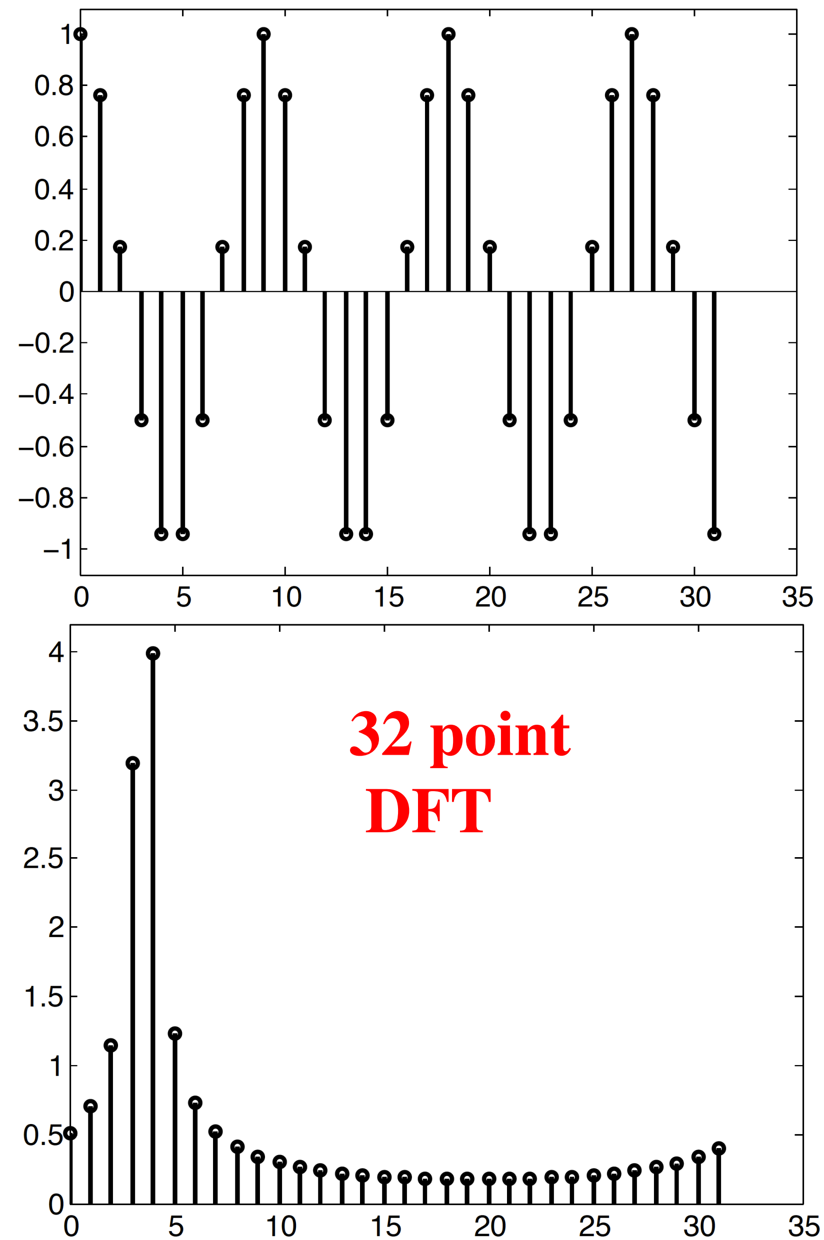
Every period is represented!

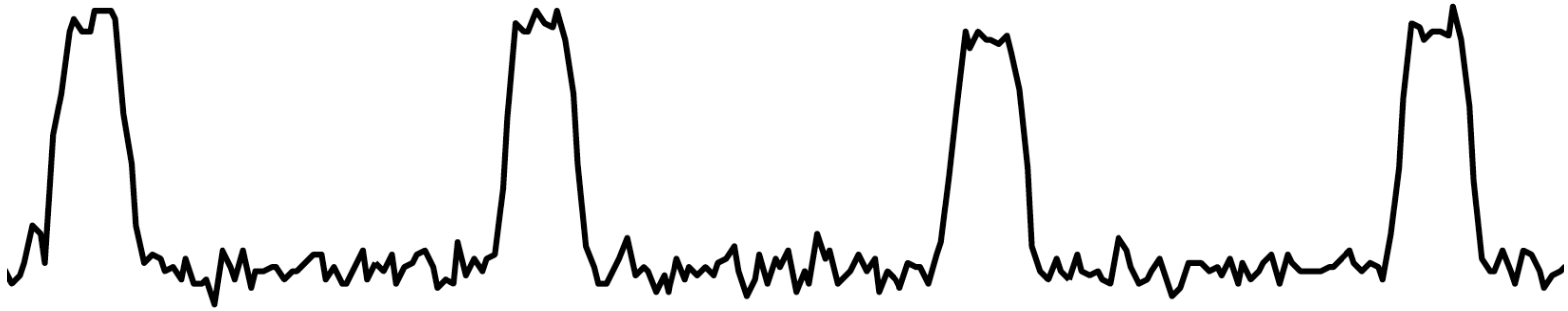
First, how much can DFT do? Sinusoid example:

$e^{j(2\pi/8)n}$ **(Re part)**



$e^{j(2\pi/9)n}$ **(Re part)**

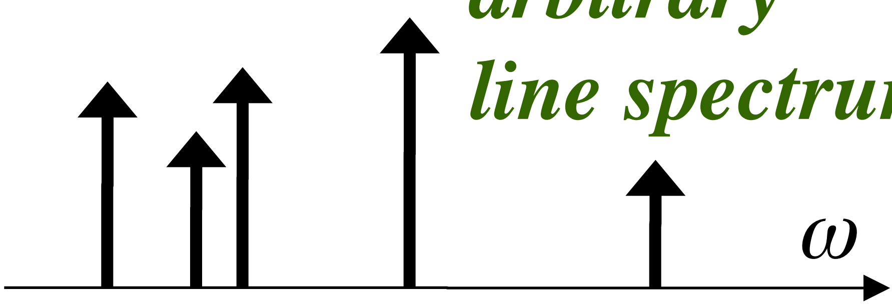




Identifying periods:

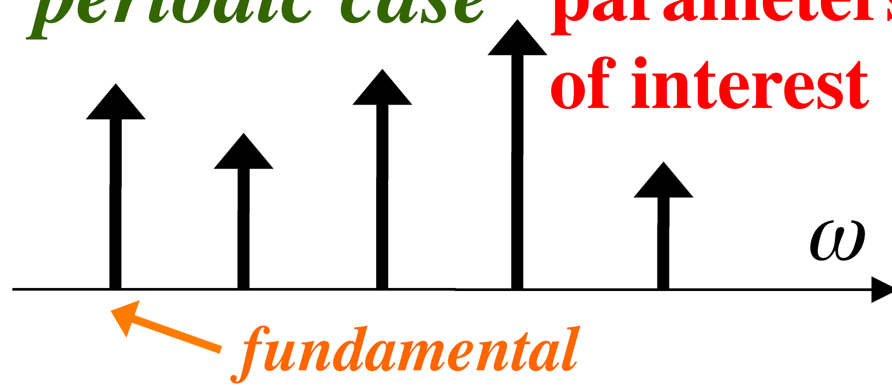
Not the same as spectrum estimation

arbitrary line spectrum



periodic case

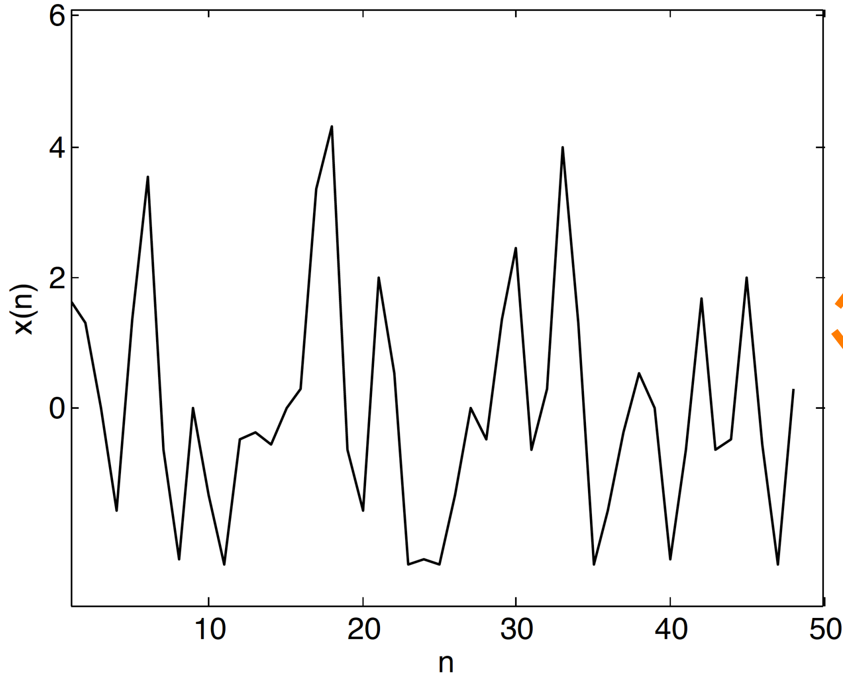
fewer parameters of interest

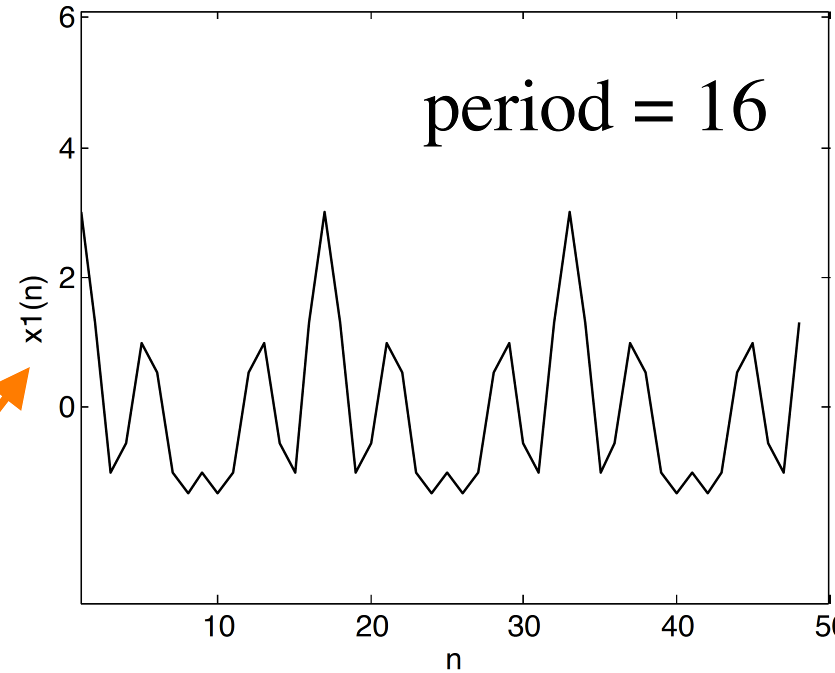


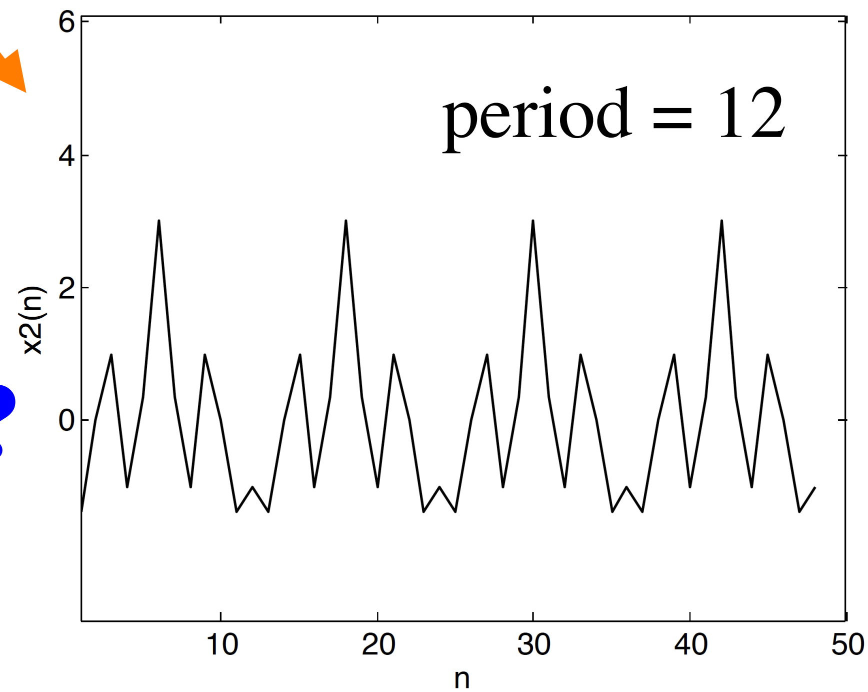
DFT, MUSIC, etc., will work, but are not necessarily best …

Ramanujan offers something new

Hidden periodic components





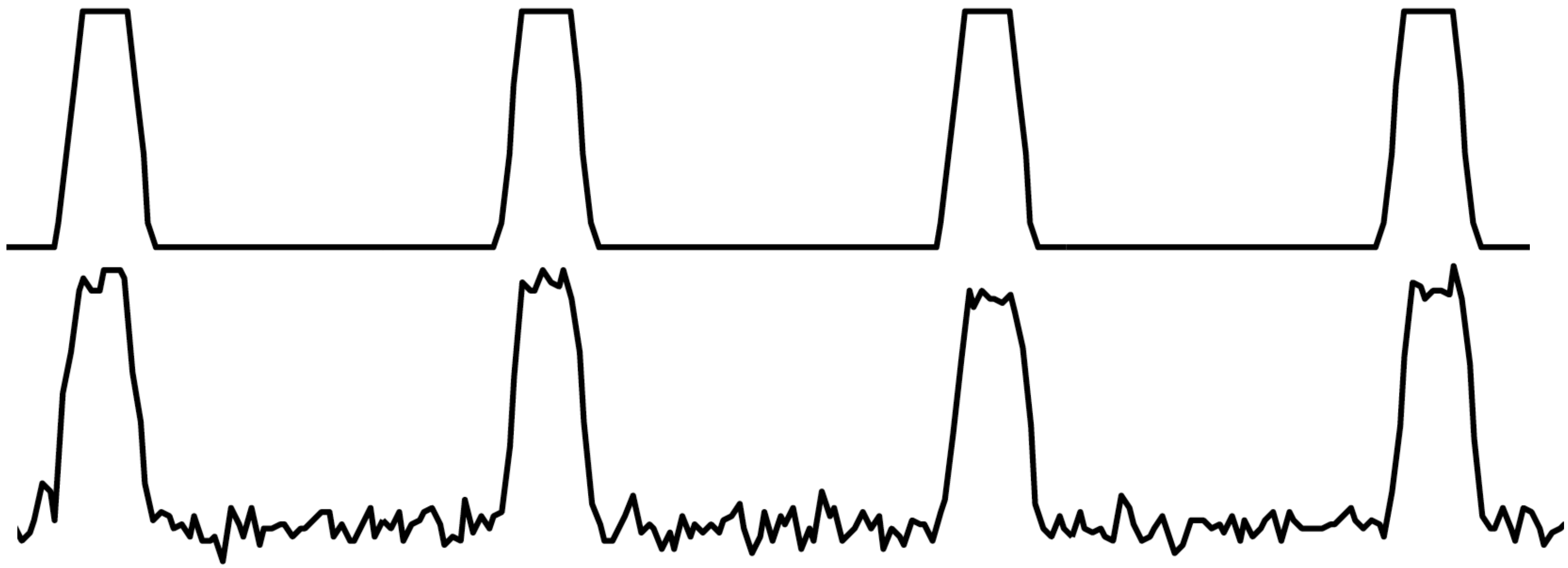


Does not "look" periodic

Sparse representation?

Ramanujan offers something new

Importance of periodic components



- Pitch identification acoustics (music, speech, …)
- Time delay estimation in sensor arrays
- Medical applications
- Genomics and proteomics
- Radar
- Astronomy

Notations

(k, q) = gcd of *k* and *q*

$(k, q) = 1$ means *k* and *q* are coprime

$W_q = e^{-j2\pi/q}$ *q*th root of unity

$\phi(q) = \#$ of integers k in $1 \leq k \leq q$ satisfying $(k, q) = 1$

= Euler's totient function

$d|q$: *d* is a divisor (or factor) of *q*

The phi-function (Euler's totient):

$\phi(q) =$ # of integers k in $1 \leq k \leq q$ satisfying $(k, q) = 1$

$q = 6 :$ $\{1, 5\}$ are coprime to 6, so $\phi(q) = 2$

$q = 5 :$ $\{1, 2, 3, 4\}$ are coprime to 5, so $\phi(q) = 4$

Ramanujan sum (1918)

sum over coprime frequencies only

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q}$$

q **= positive integer**

$c_q(n+q) = c_q(n)$ **has period exactly q**

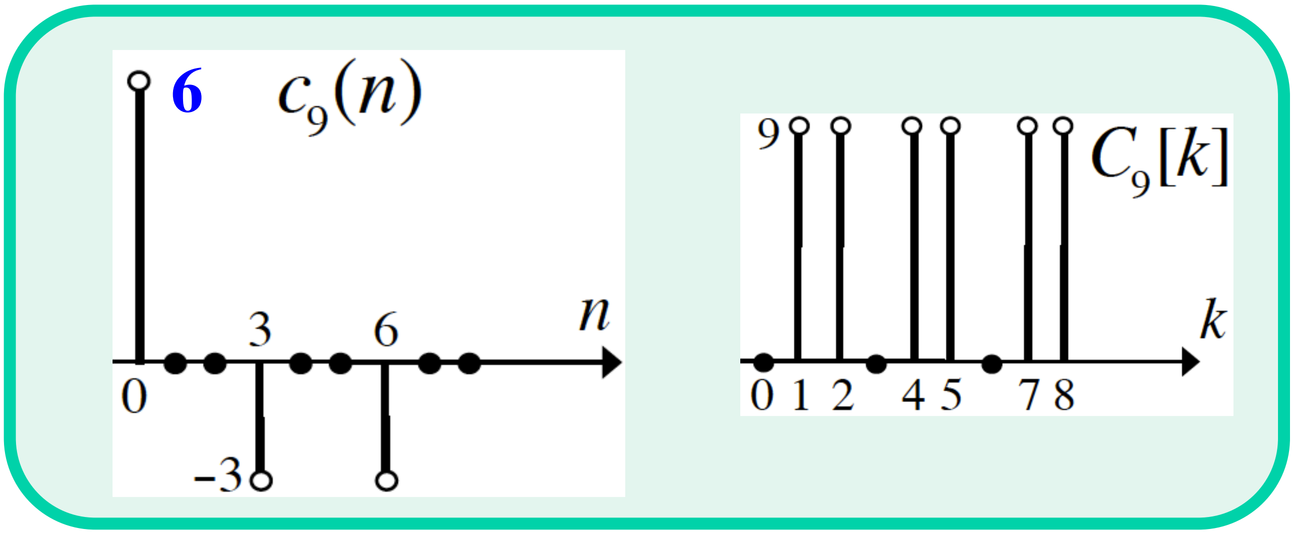
$c_q(0) = \phi(q)$ **Euler's totient**

$$\sum_{n=0}^{q-1} c_q(n) = 0, \quad \text{for } q > 1.$$

$$\sum_{n=0}^{q-1} c_q^2(n) = q\phi(q)$$

Equivalent definition using DFT

$$C_q[k] = \begin{cases} q & \text{if } (k,q) = 1 \\ 0 & \text{otherwise.} \end{cases}$$



$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q}$$

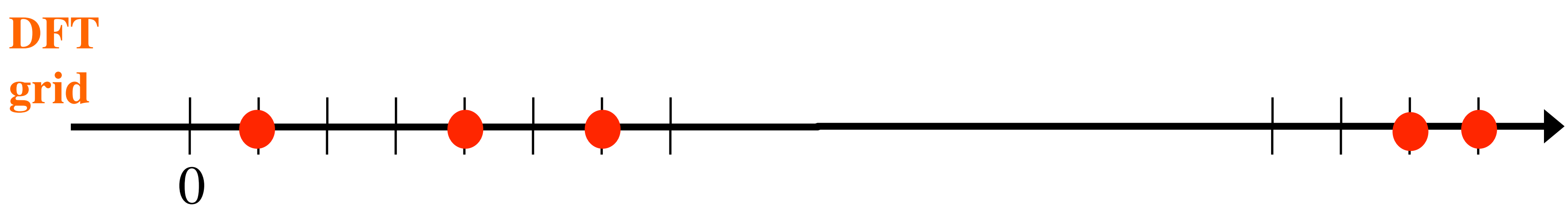
Frequency and period in Ramanujan sum

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q}$$

each term has period ***exactly*** *q*

$\phi(q)$ frequency components: $\omega_k = \dfrac{2\pi k}{q}$

primitive frequencies



Primitive frequencies, all with SAME period q

Other ways to write Ramanujan sum:

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{kn} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{-kn} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \cos\frac{2\pi kn}{q}$$

Orthogonality:

$$\sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n) = 0, \quad q_1 \neq q_2.$$

$m = \mathrm{lcm}(q_1, q_2)$

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q}$$

Theorem: Ramanujan sum is *integer* valued!

Examples:

$$\begin{aligned}
c_1(n) &= 1 \\
c_2(n) &= 1, -1 \\
c_3(n) &= 2, -1, -1 \\
c_4(n) &= 2, 0, -2, 0 \\
c_5(n) &= 4, -1, -1, -1, -1 \\
c_6(n) &= 2, 1, -1, -2, -1, 1 \\
c_7(n) &= 6, -1, -1, -1, -1, -1, -1 \\
c_8(n) &= 4, 0, 0, 0, -4, 0, 0, 0 \\
c_9(n) &= 6, 0, 0, -3, 0, 0, -3, 0, 0 \\
c_{10}(n) &= 4, 1, -1, 1, -1, -4, -1, 1, -1, 1
\end{aligned}$$

very useful from a computational perspective

Ramanujan-sum representation:

$$x(n) = \sum_{q=1}^{N} a_q c_q(n) \qquad c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j(2\pi k/q)n}$$

↑ Ramanujan sum

- What did Ramanujan do with Ramanujan sums?
- What did DSP people do with them?
- What did PP do with them?
- Does it solve the limitations of DFT?
- New directions

Ramanujan expanded arithmetic functions (1918)

Number-of-divisors: $$\sigma_0(n) = -\sum_{q=1}^{\infty} \frac{\ln q}{q} c_q(n)$$

Sum-of-divisors: $$\sigma(n) = \frac{n\pi^2}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}$$

Euler-totient: $$\phi(n) = \frac{6n}{\pi^2}\left(c_1(n) - \frac{c_2(n)}{2^2-1} - \frac{c_3(n)}{3^2-1} - \frac{c_5(n)}{5^2-1} + \frac{c_6(n)}{(2^2-1)(3^2-1)} - \frac{c_7(n)}{7^2-1} + \frac{c_{10}(n)}{(2^2-1)(5^2-1)} - \frac{c_{11}(n)}{11^2-1} - \frac{c_{13}(n)}{13^2-1} + \frac{c_{14}(n)}{(2^2-1)(7^2-1)} + \dots\right)$$

von Mangoldt function: $$\Lambda(n) = \frac{n}{\phi(n)} \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n)$$

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for prime } p \text{, with } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Some references from DSP …

[8] L. T. Mainardi, M. Bertinelli, and R. Sassi, "Analysis of T-wave alternans using the Ramanujan Transform," Computers in Cardiology, vol. 35 pp. 605-608, 2008.

[11] M. Planat, "Ramanujan sums for signal processing of low frequency noise," IEEE Intl. Frequency Control Symposium and PDA exhibition, pp. 715–720, 2002.

[12] M. Planat, M. Minarovjech, and M. Saniga, "Ramanujan sums analysis of long-periodic sequenes and $1/f$ noise," EPL journal, vol. 85, pp. 40005: 1–5, 2009.

[17] L. Sugavaneswaran, S. Xie, K. Umapathy, and S. Krishnan, "Time-frequency analysis via Ramanujan sums," IEEE Signal Processing Letters, vol. 19, pp. 352-355, June 2012.

[4] D.D. Muresan and T.W. Parks, "Orthogonal, Exactly Periodic Subspace Decomposition", IEEE Transactions on SIgnal Processing, Vol. 51, no. 9, September 2003.

References for this talk

[17] P. P. Vaidyanathan, "Ramanujan sums in the context of signal processing: Part I: fundamentals," IEEE Trans. on Signal Proc., vol. 62, no. 16, pp. 4145–4157, Aug., 2014.

[18] P. P. Vaidyanathan, "Ramanujan sums in the context of signal processing: Part II: FIR representations and applications," IEEE Trans. Sig. Proc., vol. 62, no. 16, pp. 4158–4172, Aug., 2014.

[19] P. P. Vaidyanathan, "Ramanujan subspaces and digital signal processing," Proc. Asil. Conf. Sig., Sys., and Comp., Monterey, CA, Nov. 2014.

[20] P. P. Vaidyanathan, "Multidimensional Ramanujan-sum expansions on nonseparable lattices," IEEE Intl. Conf. on Acoustics, Speech, and Signal Proc., Brisbane, Australia, April 2015.

[14] S. Tenneti and P. P. Vaidyanathan, "Nested periodic matrices and dictionaries: new signal representations for period estimation," IEEE Trans. on Sig. Proc., vol. 63, 2015.

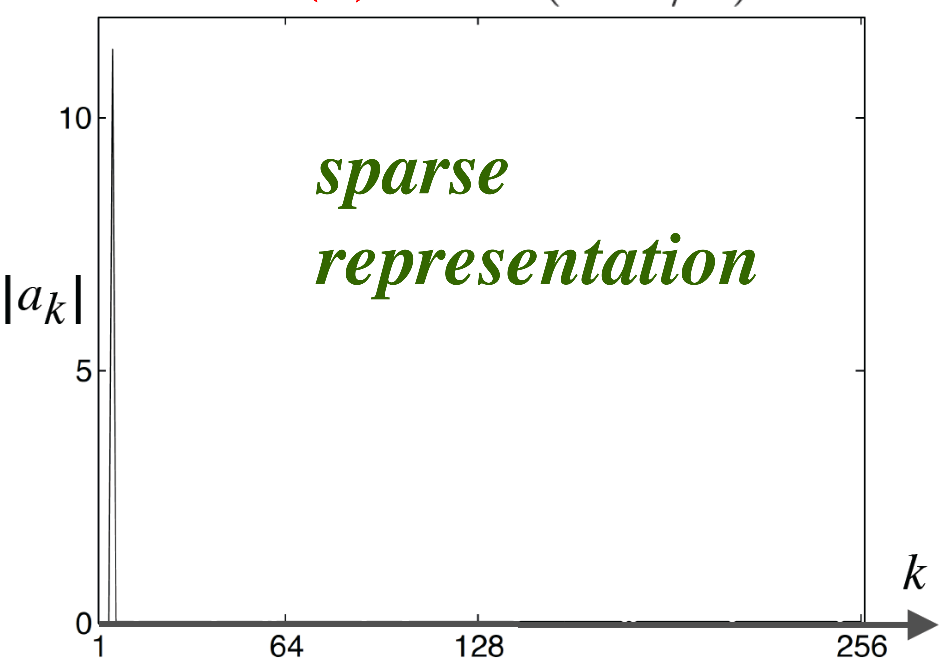
http://systems.caltech.edu/dsp/students/srikanth/Ramanujan/

An example with

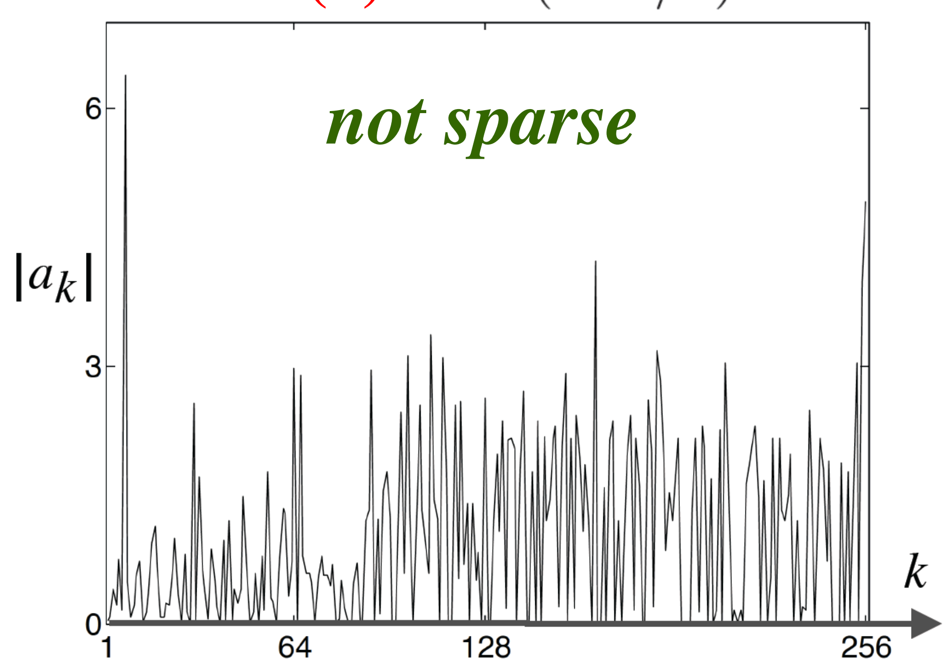
$$x(n) = \sum_{q=1}^{N} a_q c_q(n)$$

Signal duration N = 2^8 = 256

$x(n)$ = $\cos(2\pi n/6)$



$x(n)$ = $\cos(2\pi n/7)$

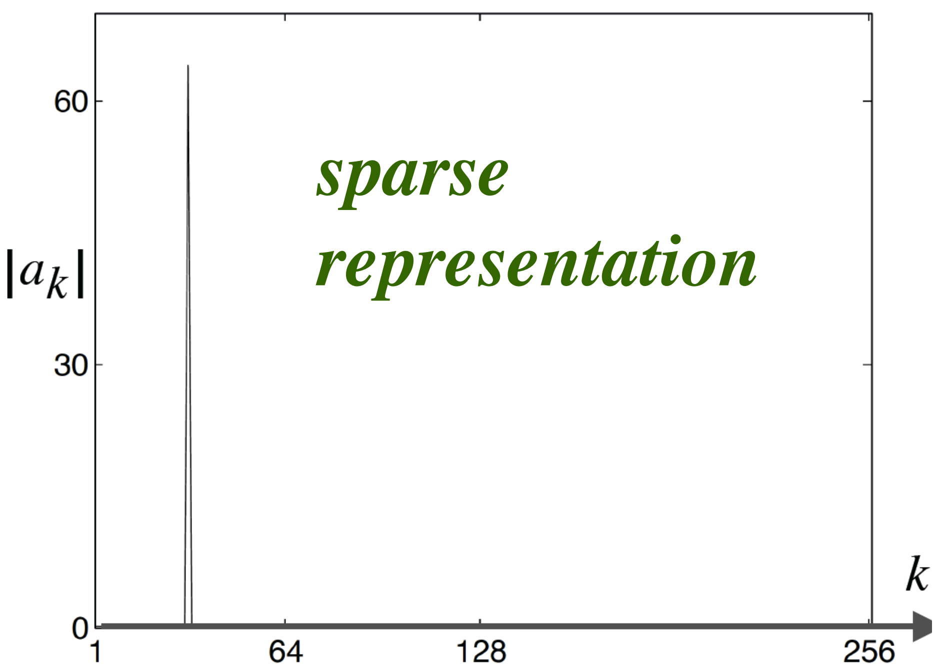


Another example with

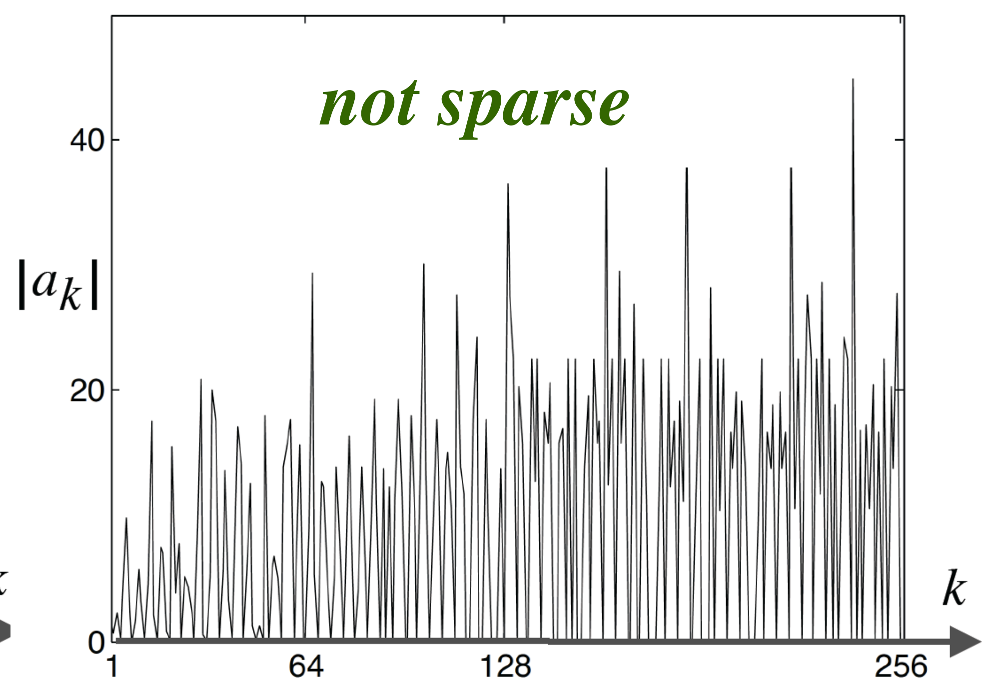
$$x(n) = \sum_{q=1}^{N} a_q c_q(n)$$

Again, signal duration $N = 2^8 = 256$

$x(n) = c_{32}(n)$



$x(n) = c_{32}(n-1)$



Reason for the problem:

$$x(n) = \sum_{q=1}^{N} a_q c_q(n)$$

each period is represented by a 1D subspace only

What to do about it?

Consider the space spanned by

$$\{c_q(n-l)\}, \quad 0 \leq l \leq q-1$$

We call this the Ramanujan subspace $\mathcal{S}_q$

Every nonzero signal in this space has period EXACTLY q

We will use the entire susbpace to represent period q components of x(n)

PPV, 2014, IEEE SP Trans.

Ramanujan subspace $\mathcal{S}_q$ is spanned by:

$$\{c_q(n-l)\}, \quad 0 \le l \le q-1$$

Theorem **(PPV, SP Trans. 2014):**

- **This space has dimension $\phi(q)$ exactly!**
- **This space is also spanned by:**

$$\{c_q(n-l)\}, \quad 0 \le l \le \phi(q)-1$$

- $\mathcal{S}_{q_1}, \ \mathcal{S}_{q_2}$ **orthogonal for** $q_1 \neq q_2$

Basis for Ramanujan $\mathcal{S}_q$ subspace

$$x(n) = \sum_{l=0}^{\phi(q)-1} \beta_l c_q(n-l)$$

real integer basis
cyclic basis

$$= \sum_{\substack{1 \le k \le q \\ (k,q)=1}} a_k e^{j2\pi kn/q}$$

complex basis
from DFT

Theorem:

Any length *N* signal can be represented as

$$x(n) = \sum_{q_i | N} \underbrace{\sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l)}_{x_{q_i}(n)}$$

periodic components in $\mathcal{S}_{q_i}$

- **Orthogonal periodic components**
- **Integer basis using Ramanujan sum**
- **Doubly indexed basis**

***PPV, SP Trans*. 2014**

Orthogonal Periodic Components

$$x(n) = \sum_{q_i|N} x_{q_i}(n)$$

$$x(n) = \sum_{q_i|N} \underbrace{\sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l)}_{x_{q_i}(n)}$$

Periodic components of x(n), period q_i

$$\mathbf{x} = \sum_{i=1}^{K} \mathbf{x}_{q_i}$$

$$\mathbf{x}_{q_i}^{\dagger} \mathbf{x}_{q_k} = 0, \quad q_i \neq q_k$$

Orthogonal projection onto S_{q_i}

Integer projection operators will find $x_{q_i}(n)$

Integer Projection Operator

$$\mathbf{B}_q = \begin{bmatrix} c_q(0) & c_q(q-1) & c_q(q-2) & \ldots & c_q(1) \\ c_q(1) & c_q(0) & c_q(q-1) & \ldots & c_q(2) \\ c_q(2) & c_q(1) & c_q(0) & \ldots & c_q(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_q(q-2) & c_q(q-3) & c_q(q-4) & \ldots & c_q(q-1) \\ c_q(q-1) & c_q(q-2) & c_q(q-3) & \ldots & c_q(0) \end{bmatrix}$$

$q \times q$

circulant

Define $\mathbf{P}_q = \dfrac{\mathbf{B}_q}{q}$ **;**

Theorem:

$\mathbf{P}_q$ **= orthogonal projection matrix onto** $\mathcal{S}_q$

Summary of the decomposition

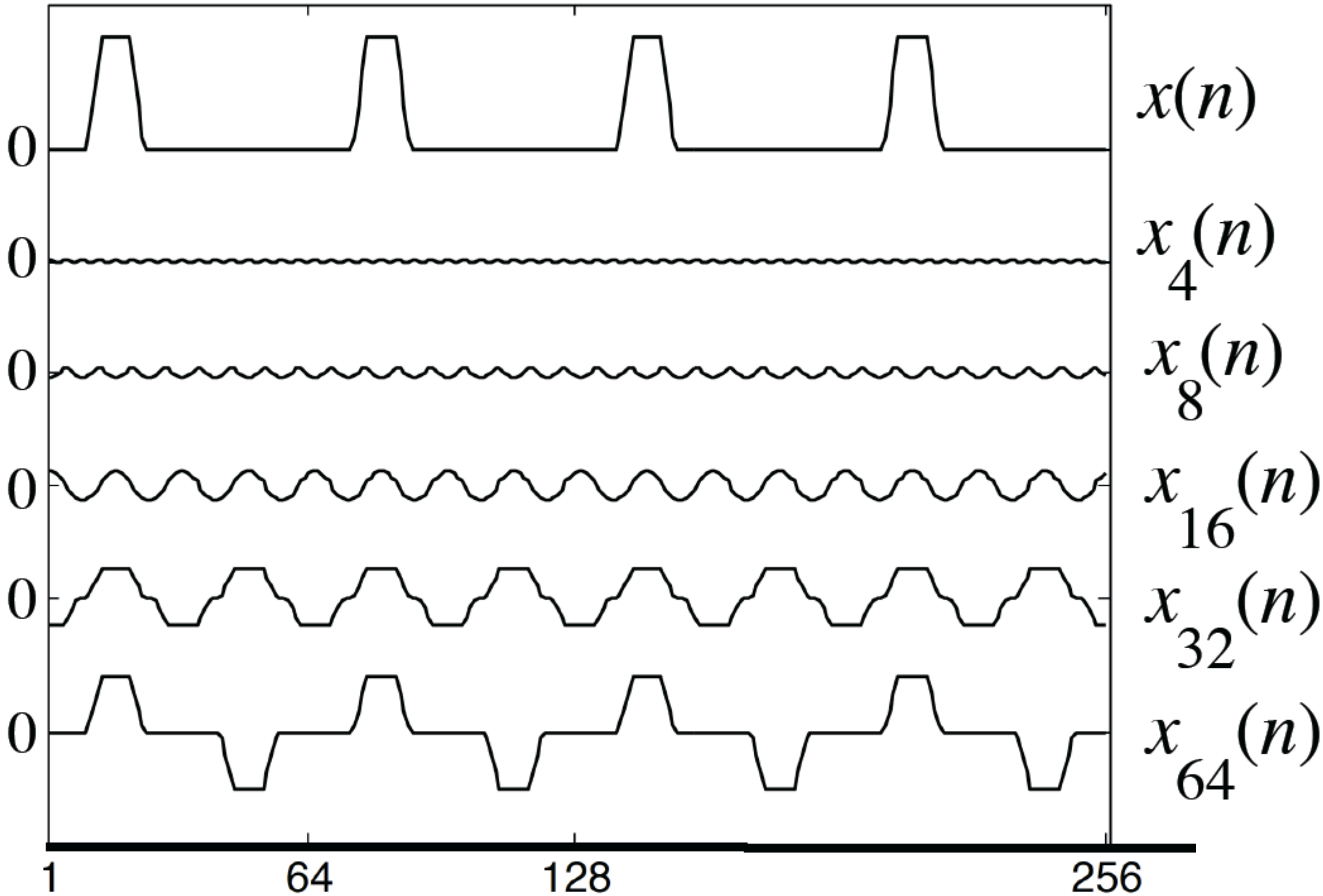
$$x(n) = \sum_{q_i|N} x_{q_i}(n) = \sum_{q_i|N} \underbrace{\sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l)}_{x_{q_i}(n) \text{ in } \mathcal{S}_{q_i}}$$

$$\sum_{n=0}^{N-1} x_{q_i}(n) x^*_{q_k}(n) = 0$$ **orthogonality**

$$\sum_{n=0}^{N-1} x_{q_i}(n) x^*_{q_k}(n-l) = 0$$ **total orthogonality**

$x_{q_i}(n)$ **has period exactly** q_i

Example of Ramanujan space projections



Periodic, orthogonal, projections

Theorem (LCM property):

Consider a sum $x(n) = \sum_{m=1}^{K} x_{q_m}(n)$

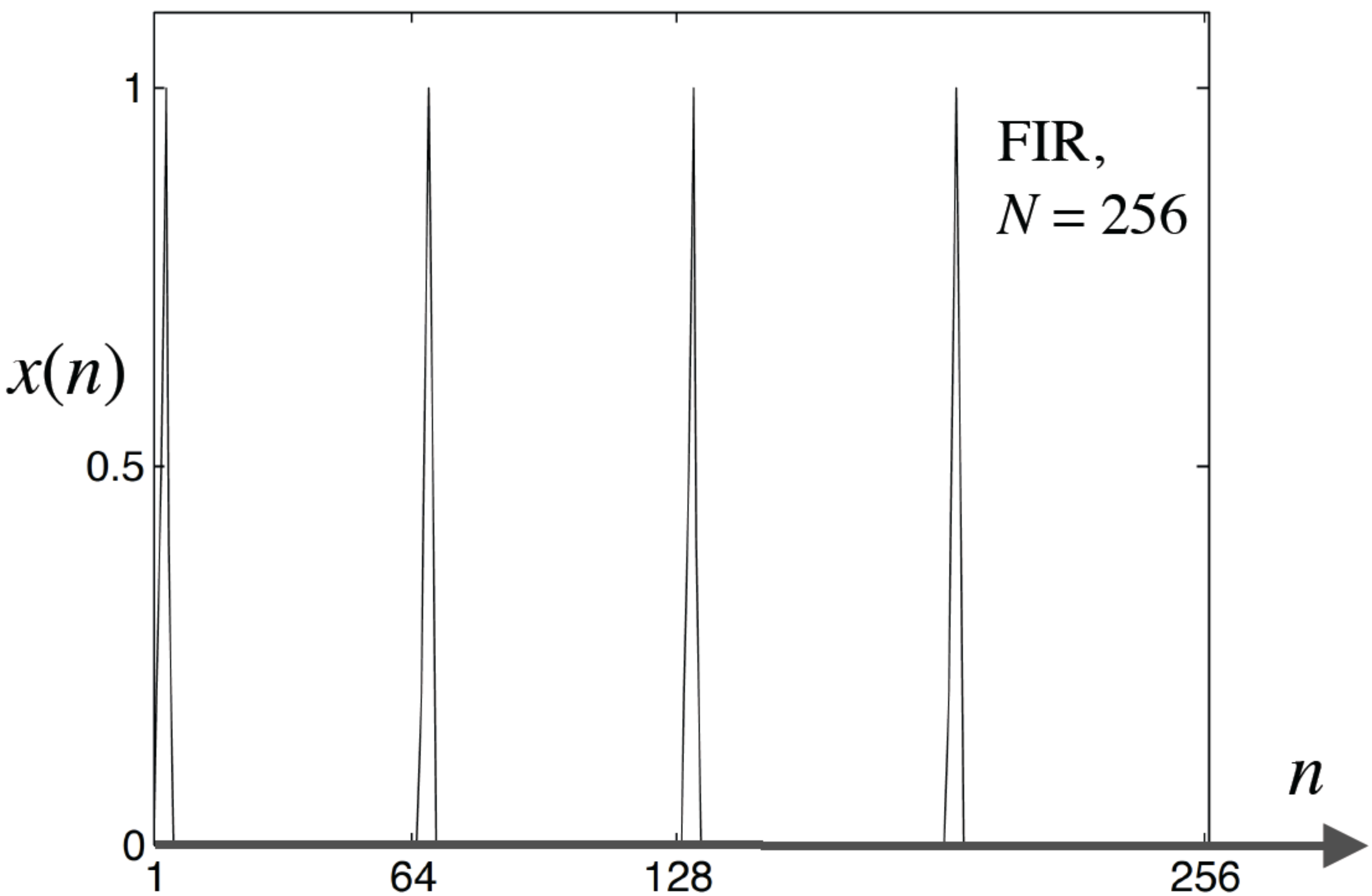
where $x_{q_m}(n) \in \mathcal{S}_{q_m}$

This has period $= \text{lcm}(q_1, q_2, \ldots, q_K)$

(can't be smaller).

Recall signals in $\mathcal{S}_q$ **have period *q*** **(can't be smaller).**

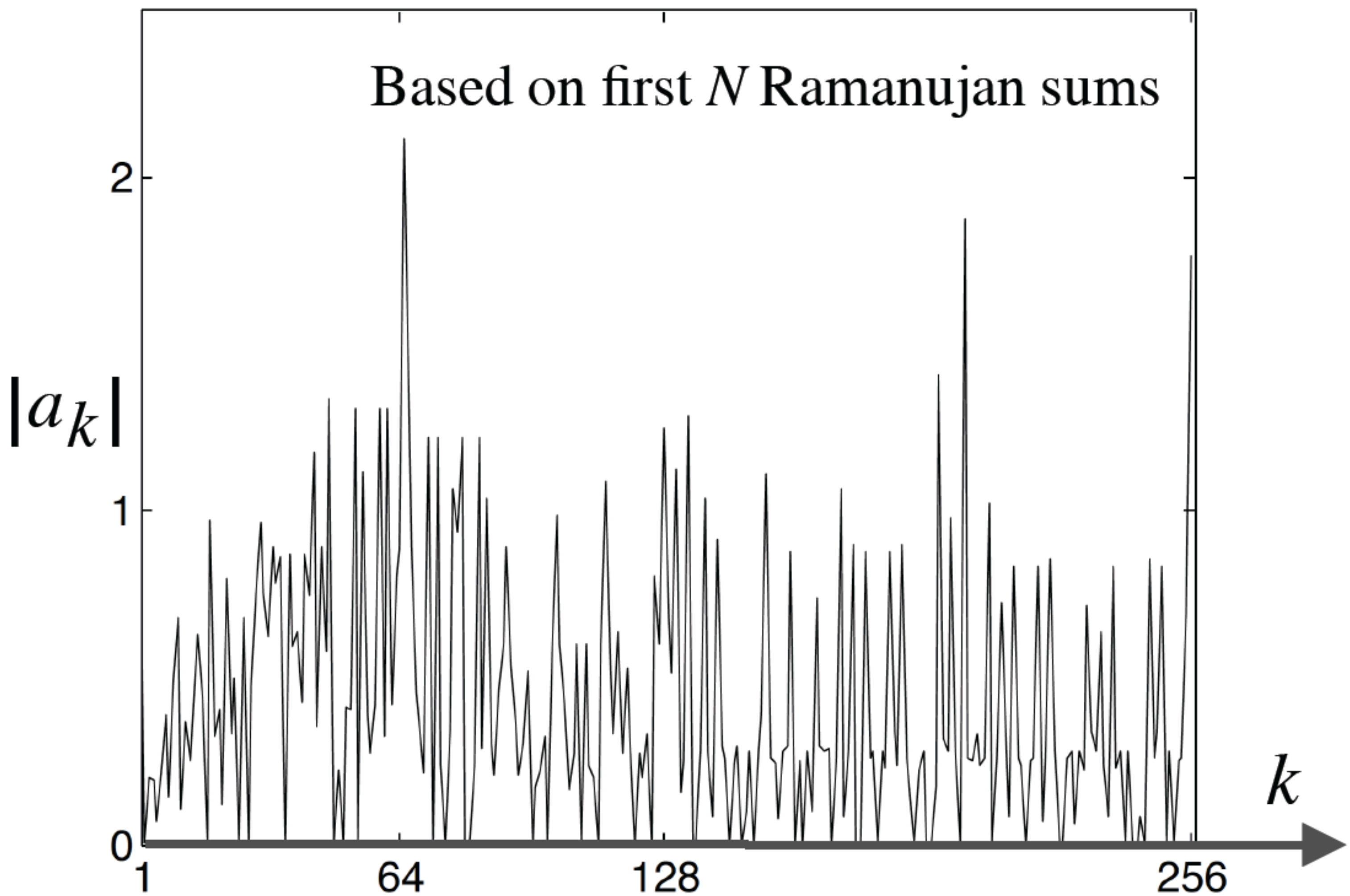
Example 1



periodic input, period = 64

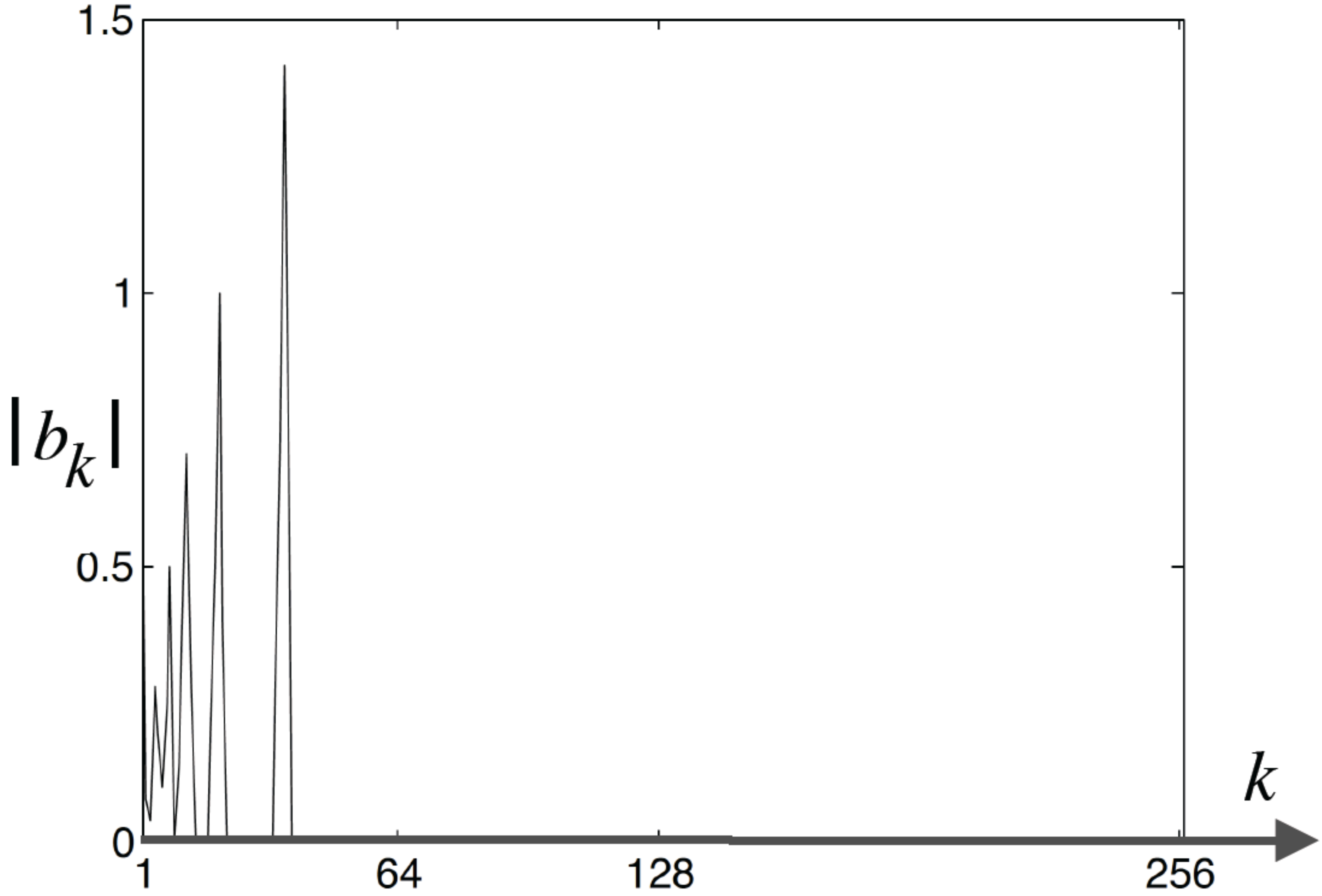
Method 1:

$$\underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\mathbf{x}} = \mathbf{A}_N \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}}_{\mathbf{a}}$$

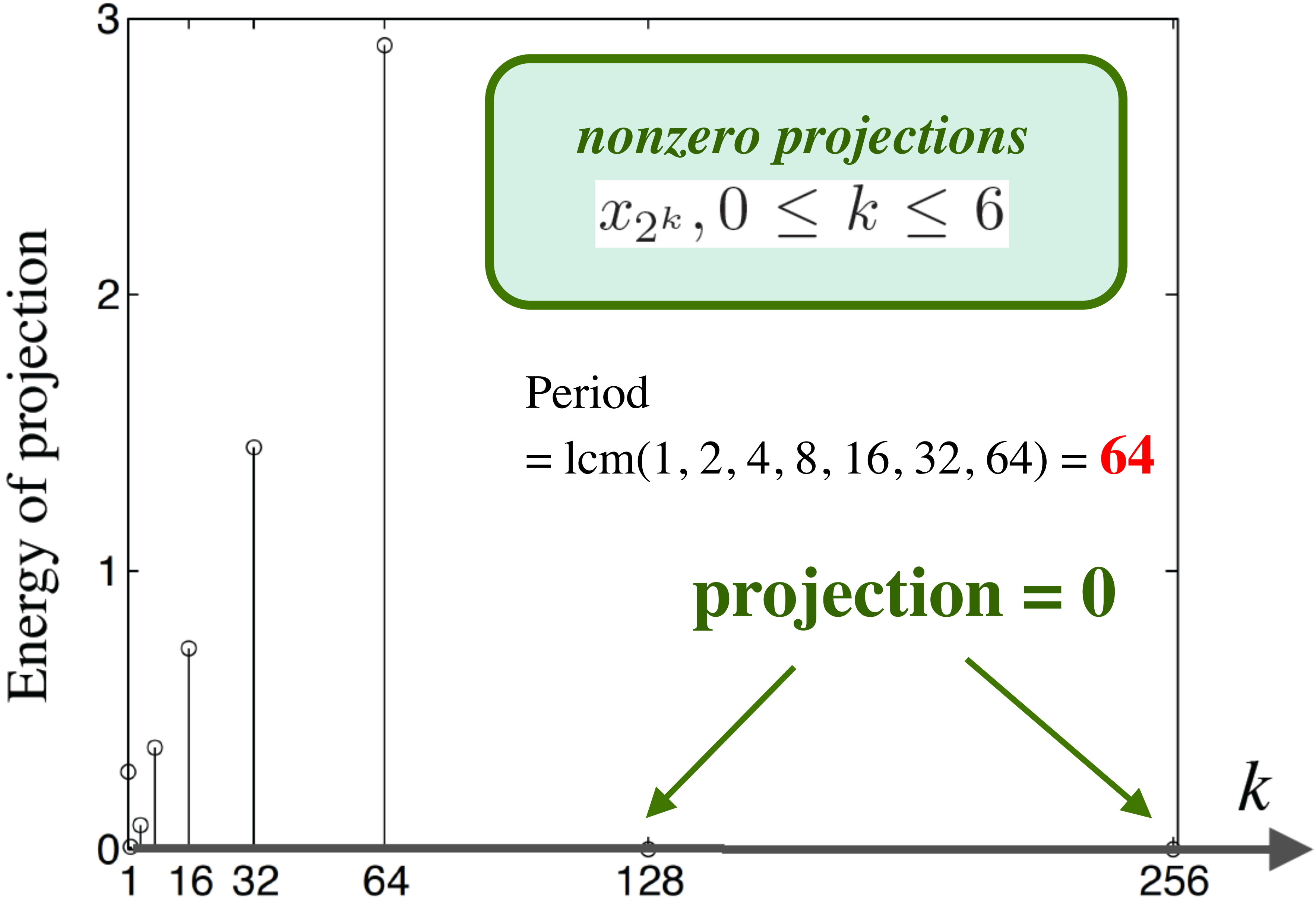


Method 2: $\mathbf{x} = \mathbf{F}_N \mathbf{b}$

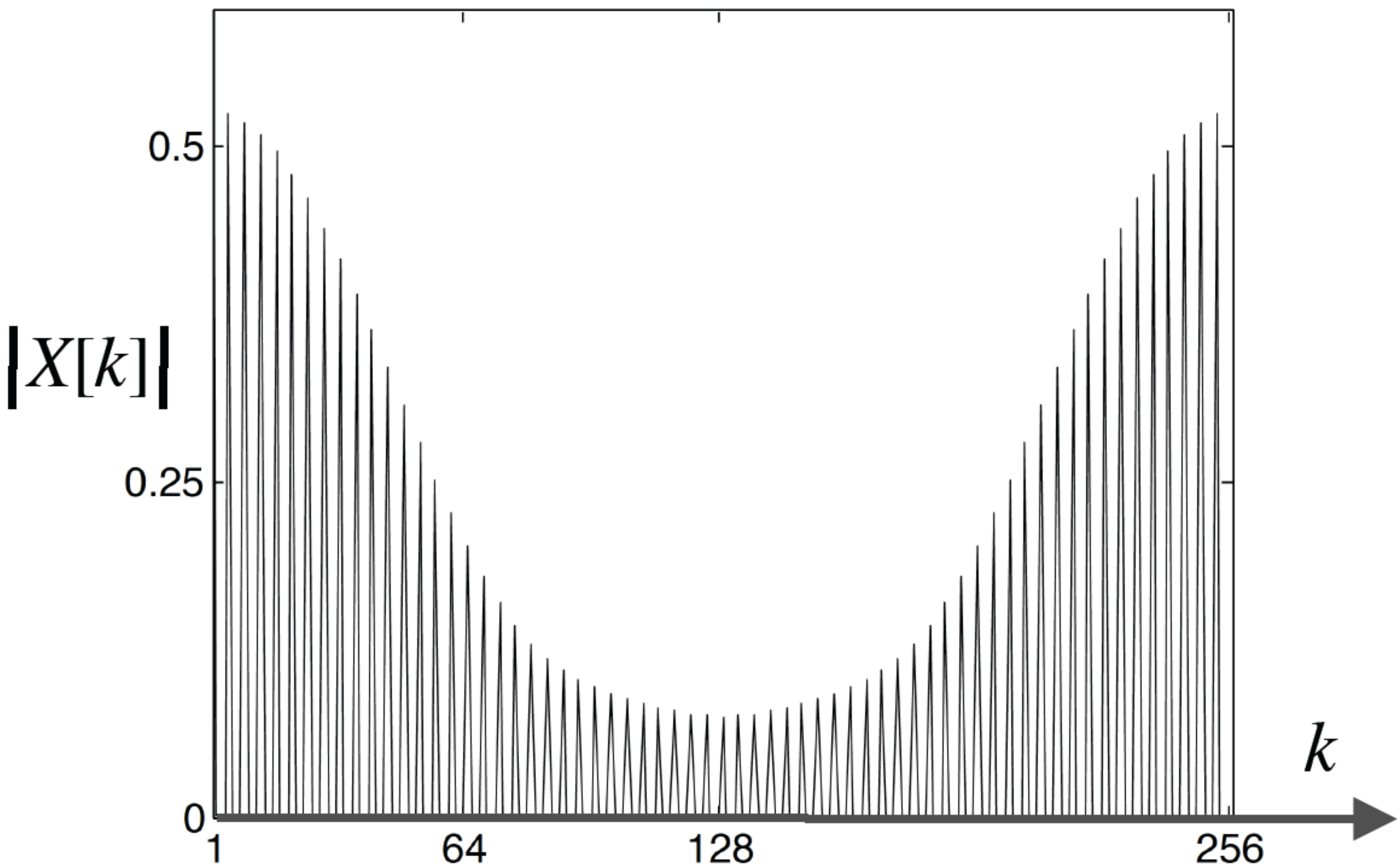
$$x(n) = \sum_{q_i | N} \underbrace{\sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l)}_{x_{q_i}(n)}$$



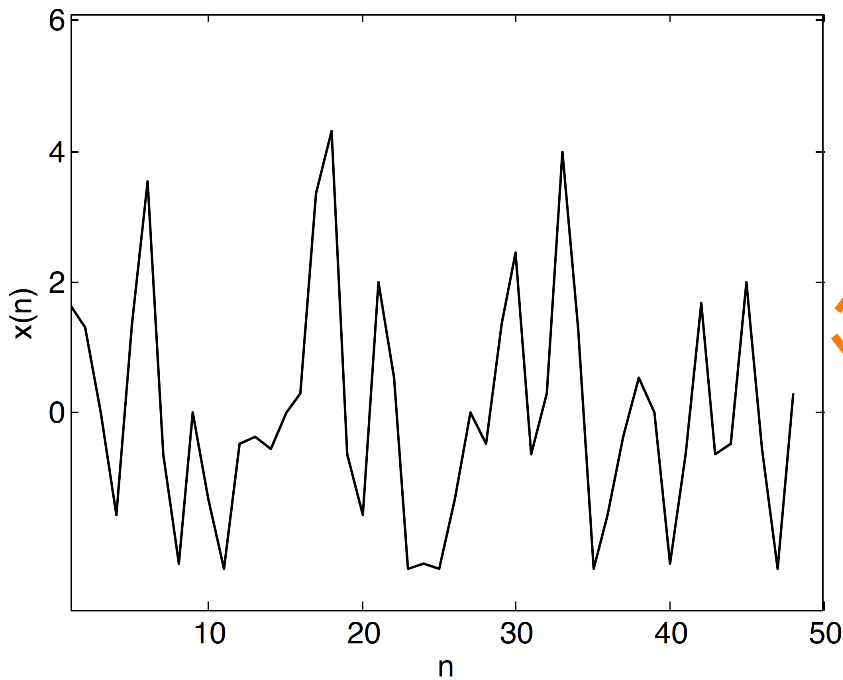
Projections onto Ramanujan spaces S_{q_i}



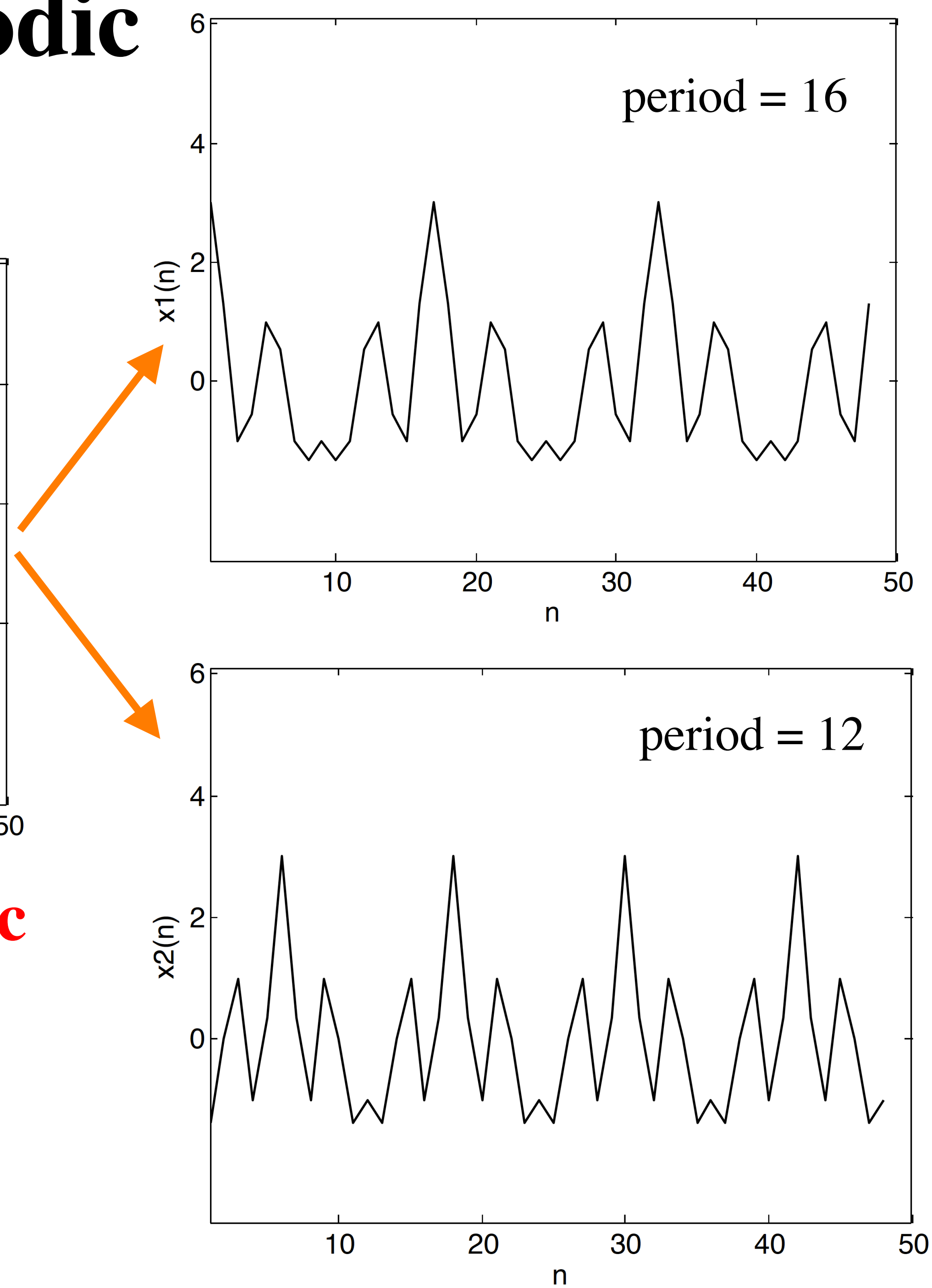
DFT plot



Ex. 2: Hidden periodic components

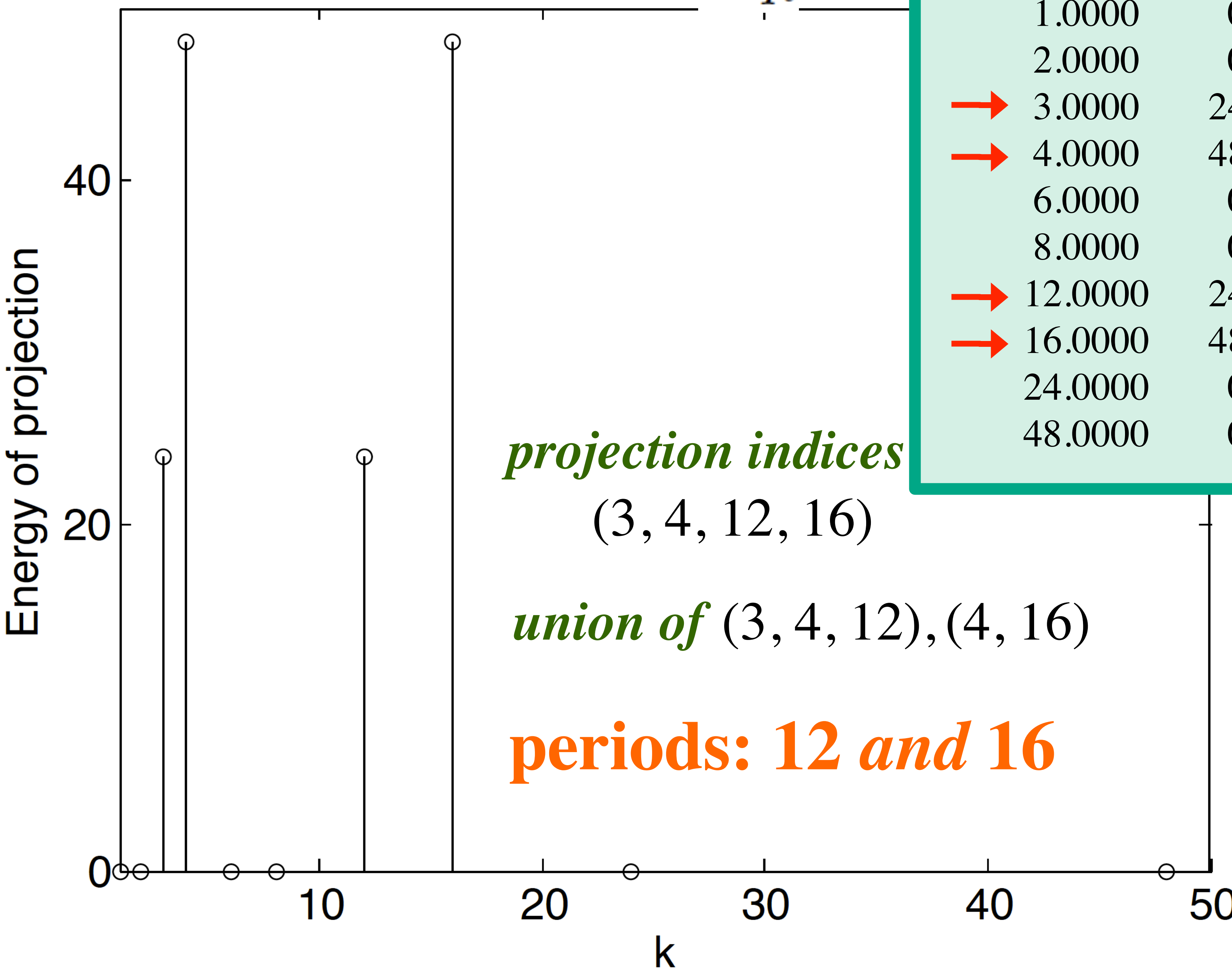


Does not "look" periodic



Projections onto spaces $\mathcal{S}_{q_i}$

index k	*energy*
1.0000	0.0000
2.0000	0.0000
→ 3.0000	24.0000
→ 4.0000	48.0000
6.0000	0.0000
8.0000	0.0000
→ 12.0000	24.0000
→ 16.0000	48.0000
24.0000	0.0000
48.0000	0.0000



projection indices
$(3, 4, 12, 16)$

union of $(3, 4, 12), (4, 16)$

periods: 12 *and* 16

More stuff …

- **Dictionary methods**
- **2D versions**
- **Time-period plane**
- **Ramanujan filter banks**
- **Medical applications**
- **DNA microsatellites, proteins**

Dictionaries for periodicity

Ramanujan dictionary

$$\Phi(N) \triangleq \sum_{m=1}^{N} \phi(m)$$

$$\mathbf{A} = \left[\begin{array}{c|c|cc|cc|cccc} 1 & 1 & 2 & -1 & 2 & 0 & 4 & -1 & -1 & -1 \\ 1 & -1 & -1 & 2 & 0 & 2 & -1 & 4 & -1 & -1 \\ 1 & 1 & -1 & -1 & -2 & 0 & -1 & -1 & 4 & -1 \\ 1 & -1 & 2 & -1 & 0 & -2 & -1 & -1 & -1 & 4 \\ 1 & 1 & -1 & 2 & 2 & 0 & -1 & -1 & -1 & -1 \end{array}\right]$$

2*N*

$\phi(3)$ $\phi(4)$ $\phi(5)$

$$\mathbf{x} = \mathbf{Ay}$$

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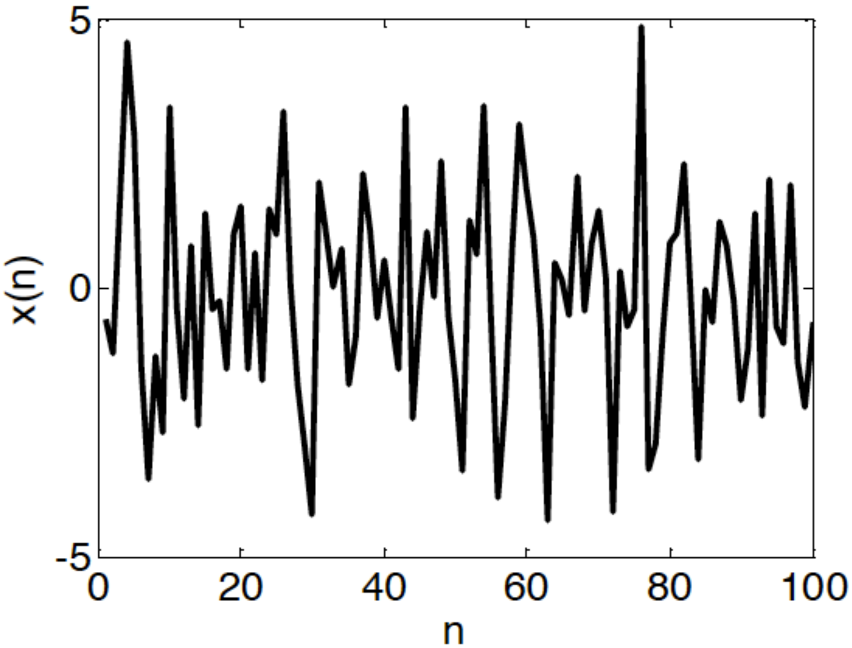
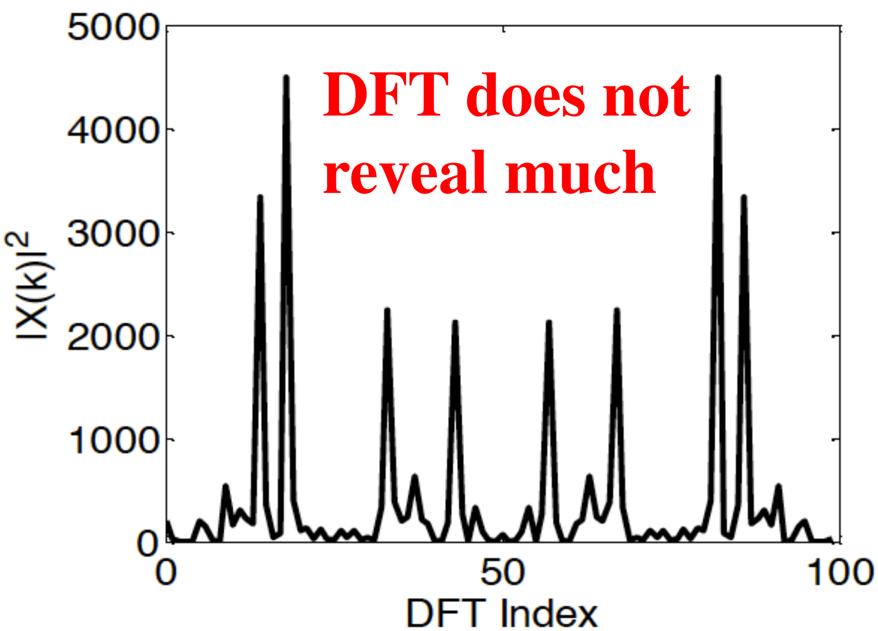
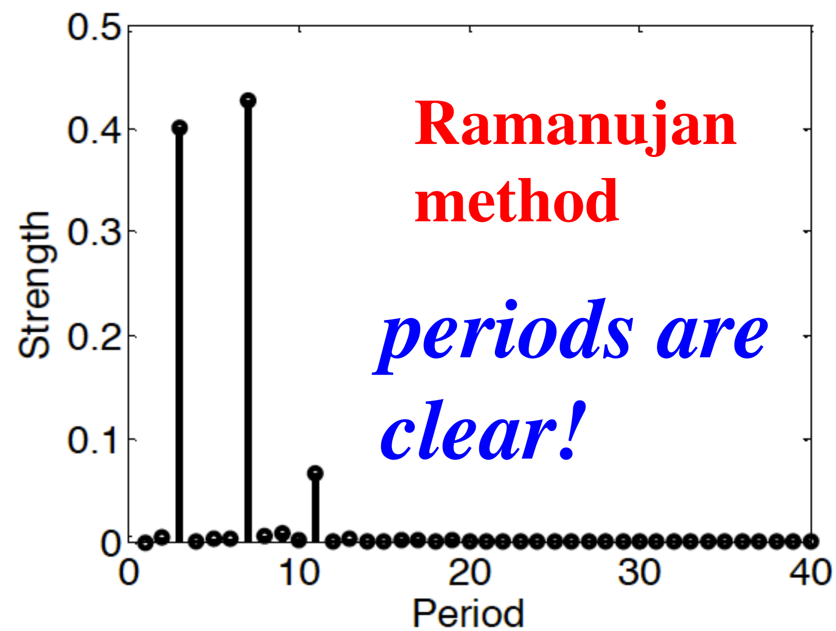
Any period < N can be identified



Hidden periods: 3, 7, 11





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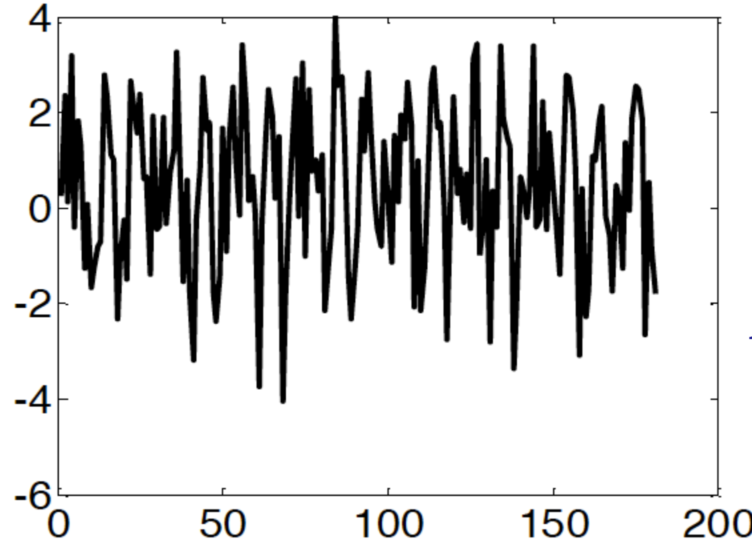
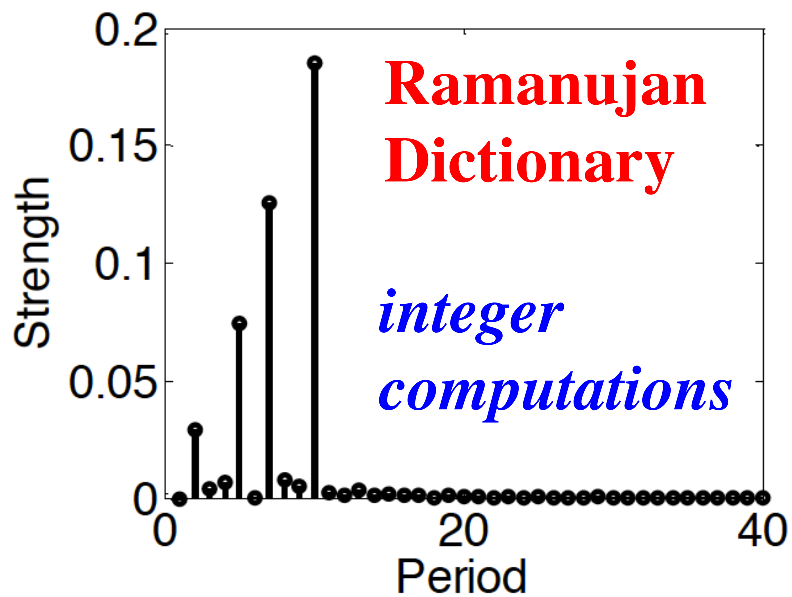
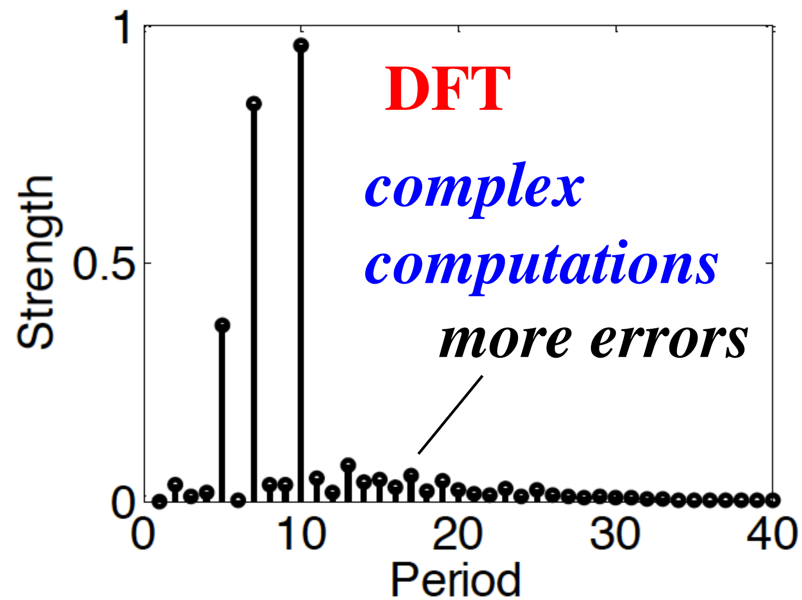


Hidden periods: 7, 10





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Min. # of samples for period estimation

Assume period is known to belong to this set:

$$\mathbb{P} = \{1, 2, 3, \ldots, P_{max}\}$$

Theoretical bound:

$$L_{min} = \max_{P_i, P_j \in \mathbb{P}} P_i + P_j - \gcd(P_i, P_j)$$

Theoretical bound, *N* hidden periods

$$M_{min} = \max_{\substack{\mathbb{P}_i, \mathbb{P}_j \subset \mathbb{P} \\ \mathbb{P}_i, \mathbb{P}_j \text{ are} \\ \text{M}-\text{sets of size } = N}} \sum_{d \in \text{D.S.}(\{\mathbb{P}_i \cup \mathbb{P}_j\})} \phi(d)$$

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Assume period is known to belong to this set:

$$\mathbb{P} = \{1, 2, 3, \ldots, P_{max}\}$$

Dictionary method:

$$L \geq L_{min}^{(D)} = 2P_{max}$$

Dictionary method, *N* hidden periods

$$\mathbb{P} = \{1, 2, 3, \ldots, P_{max}\}$$

$$M_{min}^{(D)} = \max_{\substack{\mathbb{P}_i \subseteq \mathbb{P} \\ \mathbb{P}_i \text{ is an} \\ \text{M-set of size N}}} 2 \times \sum_{d \in \text{D.S.}(\{\mathbb{P}_i\})} \phi(d)$$

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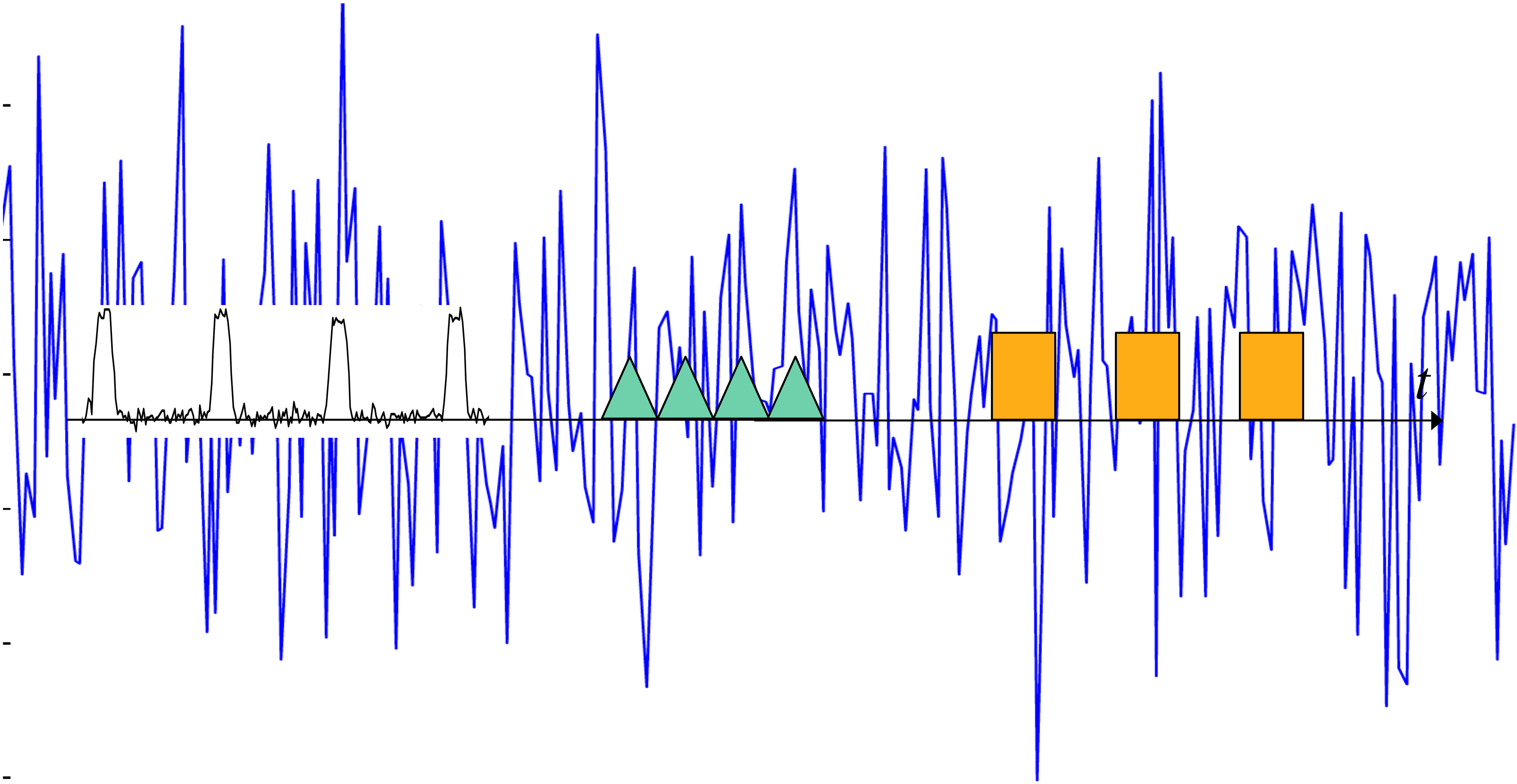
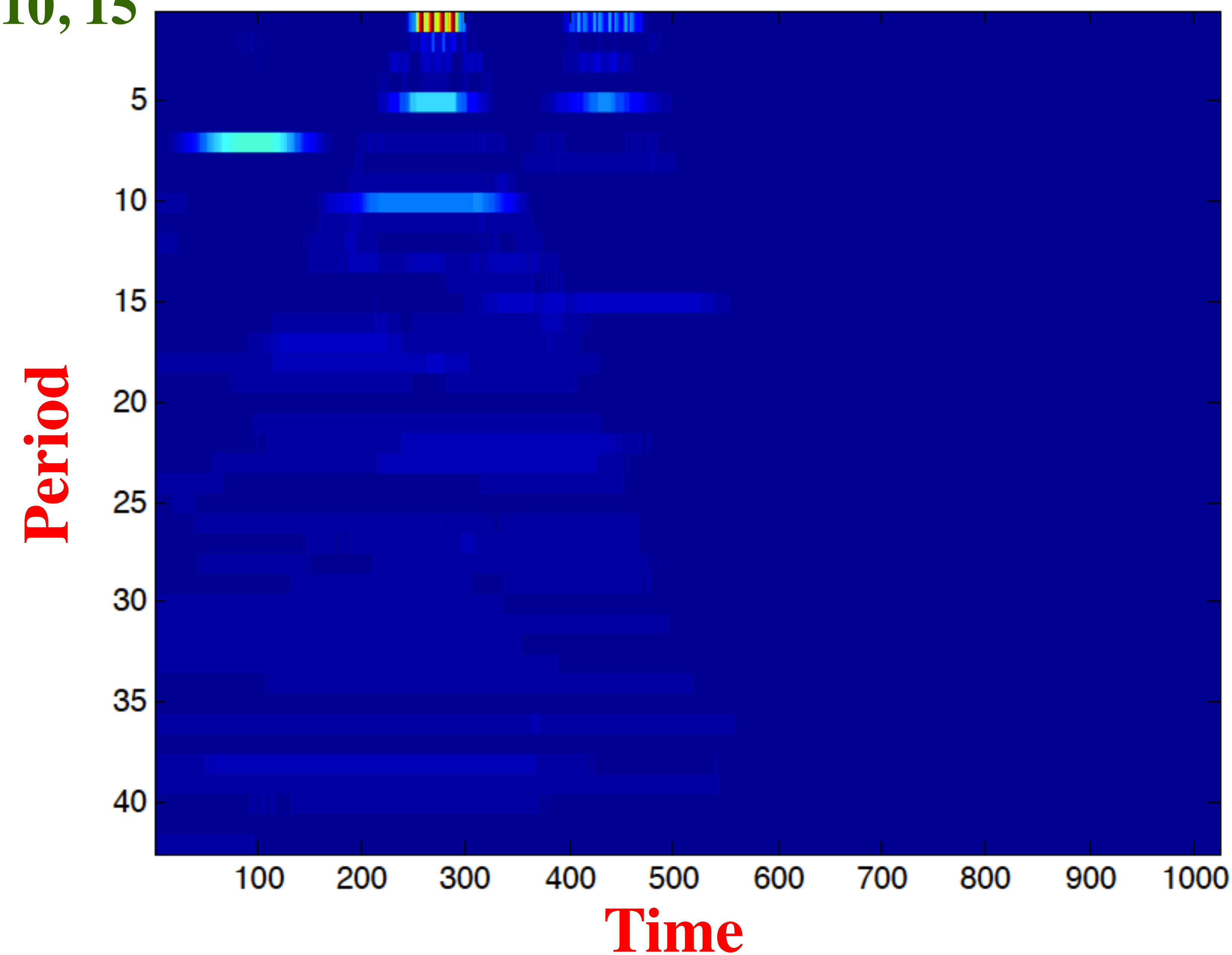
Time-localization

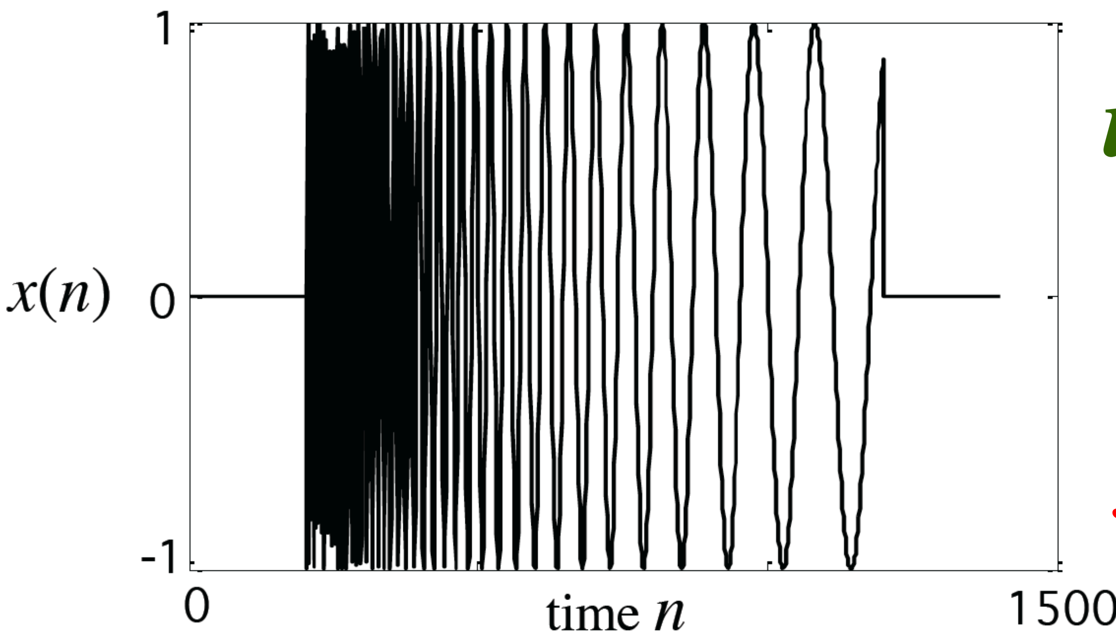


Need a more fundamental approach to time-period plane

7, 10, 15



S. Tenneti, P. P. Vaidyanathan, ICASSP **2015**



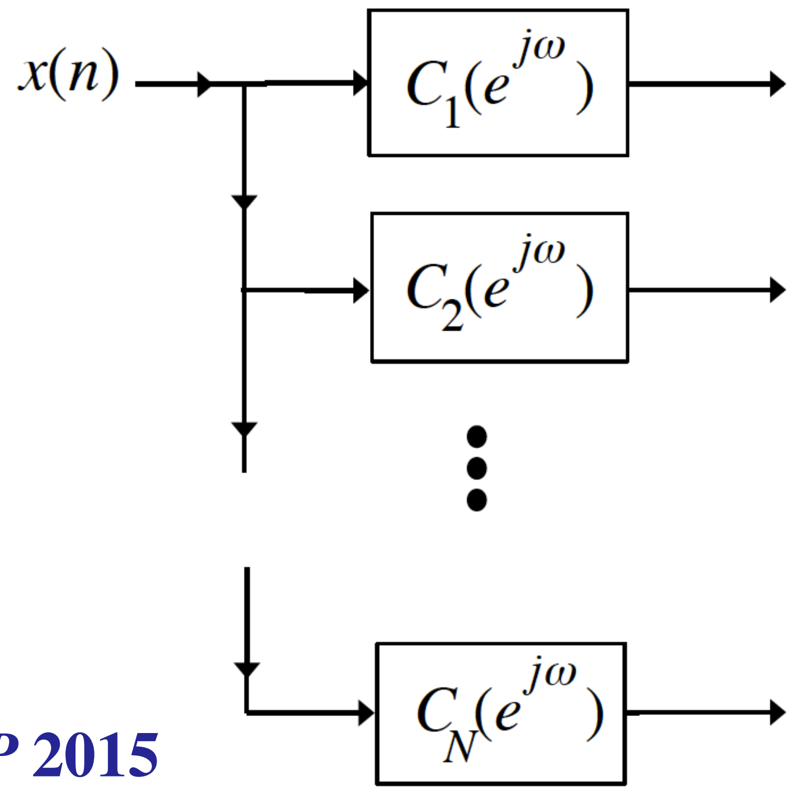
inverse chirp signal

Track period as a function of time

$$x_c(t) = \sin(1/at)$$

$$a = 0.01/\pi$$

sample spacing $T = 0.005s$



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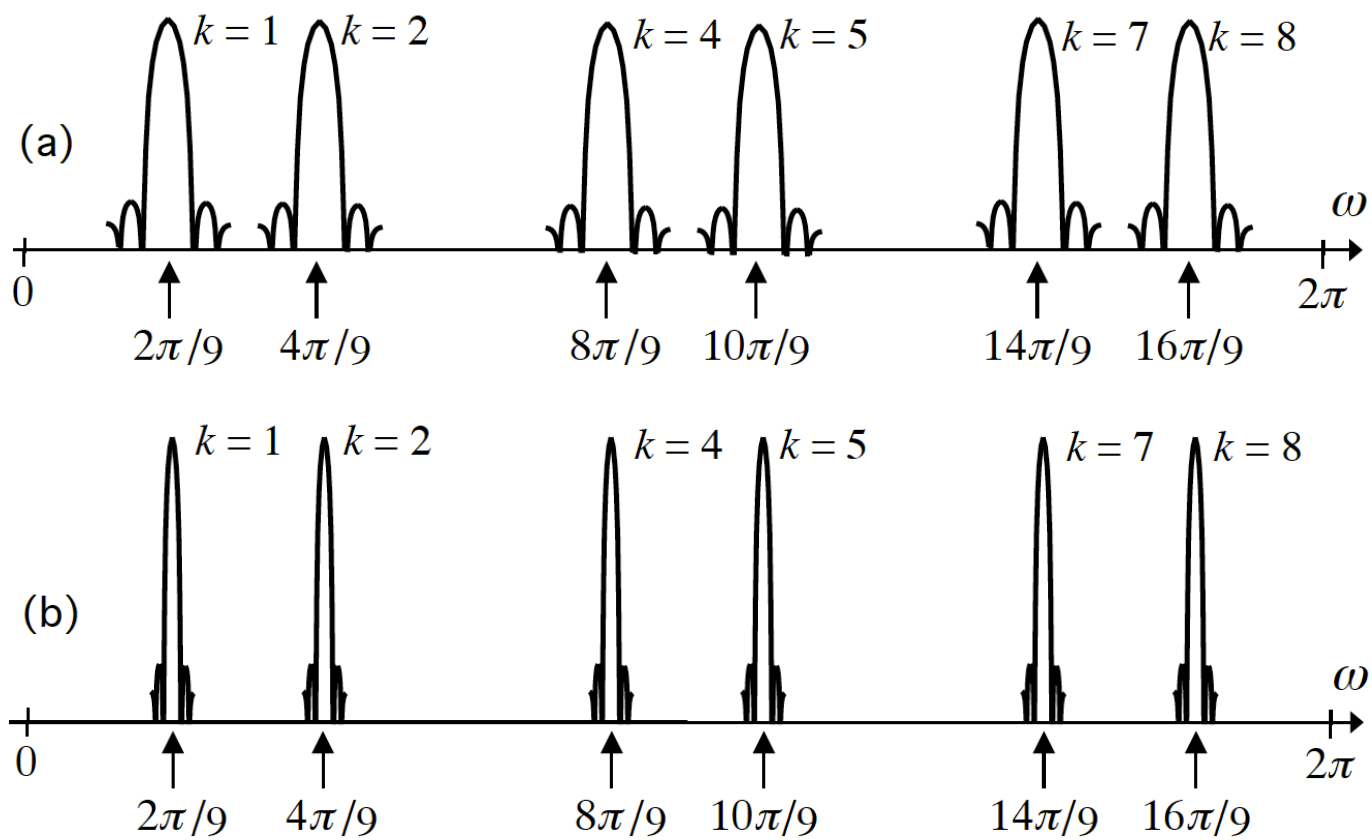
Can use FIR filter banks in practice:

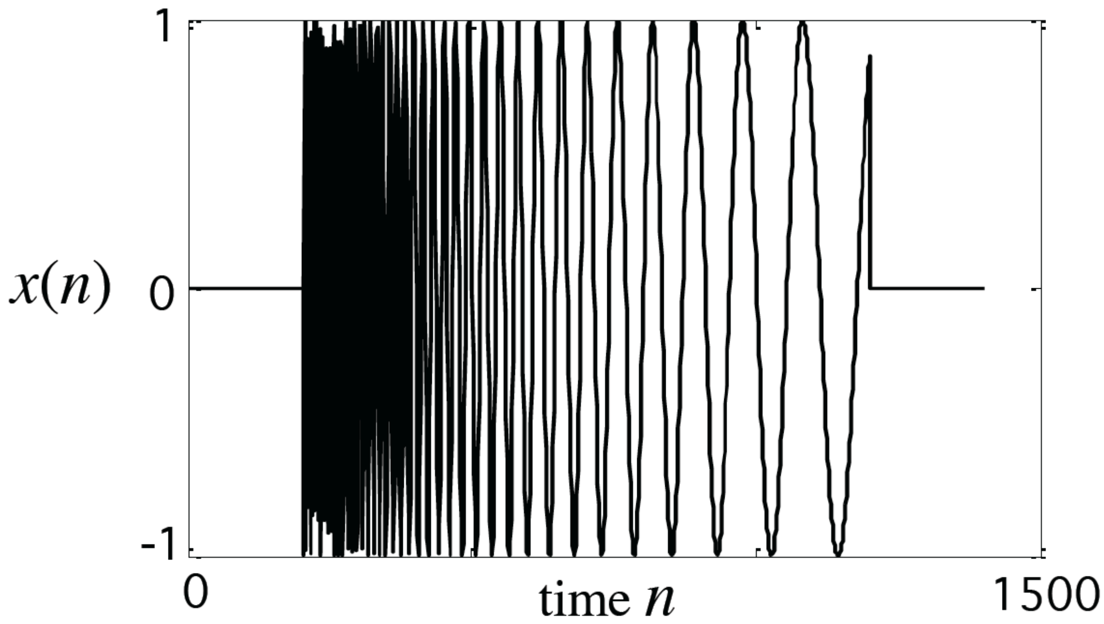
$$C_q^{(l)}(z) = \sum_{n=0}^{lq-1} c_q(n) z^{-n}$$

Ex: $q = 9$

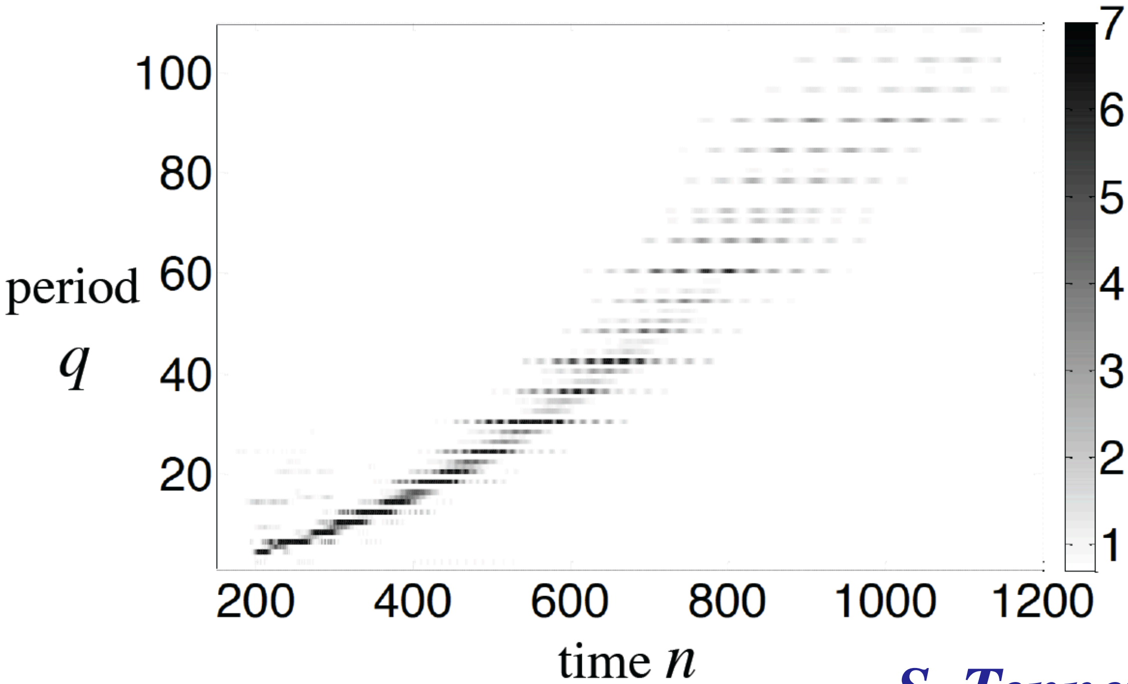


small l

large l



chirp signal



Ramanujan filter bank output

$l = 5$

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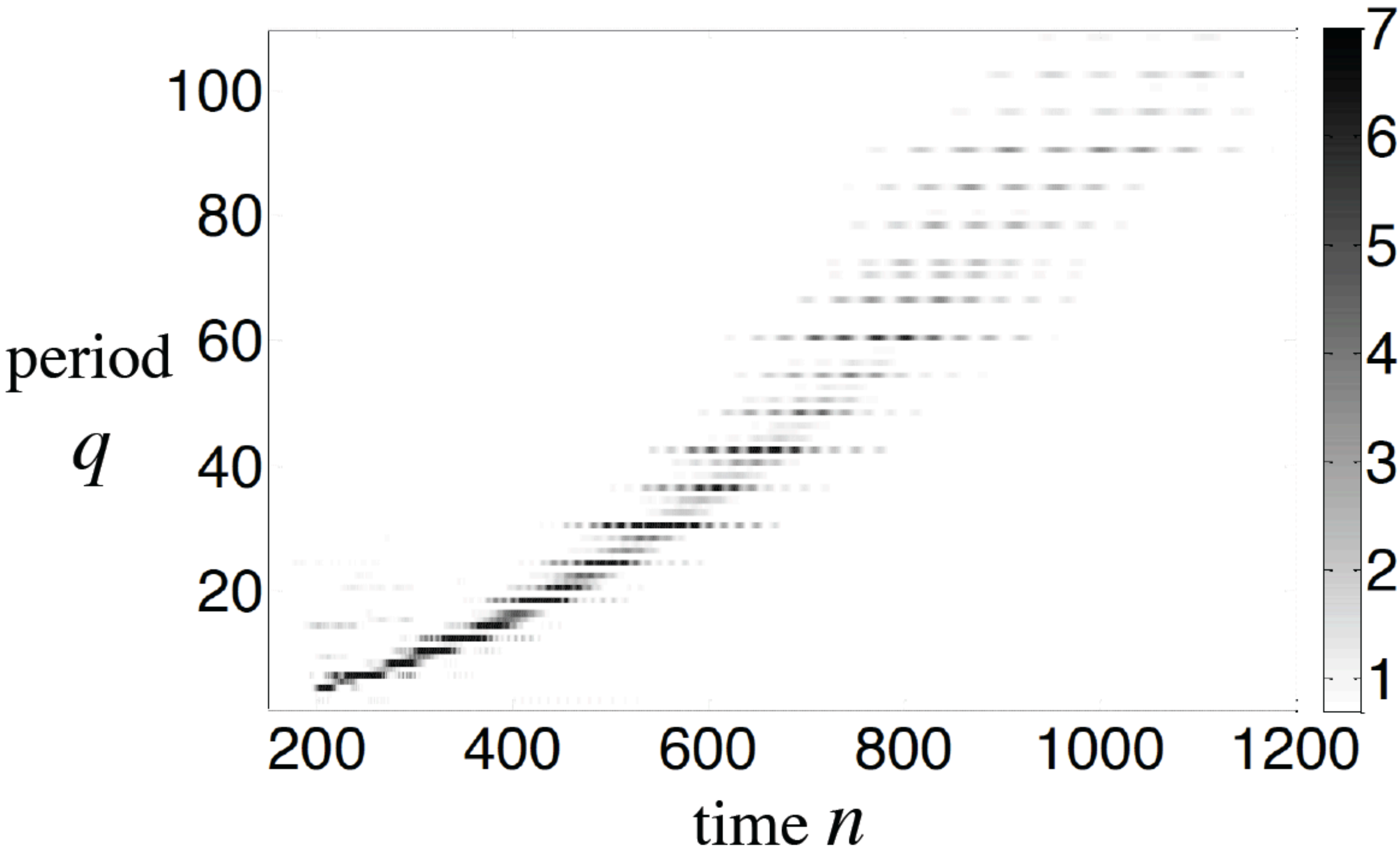
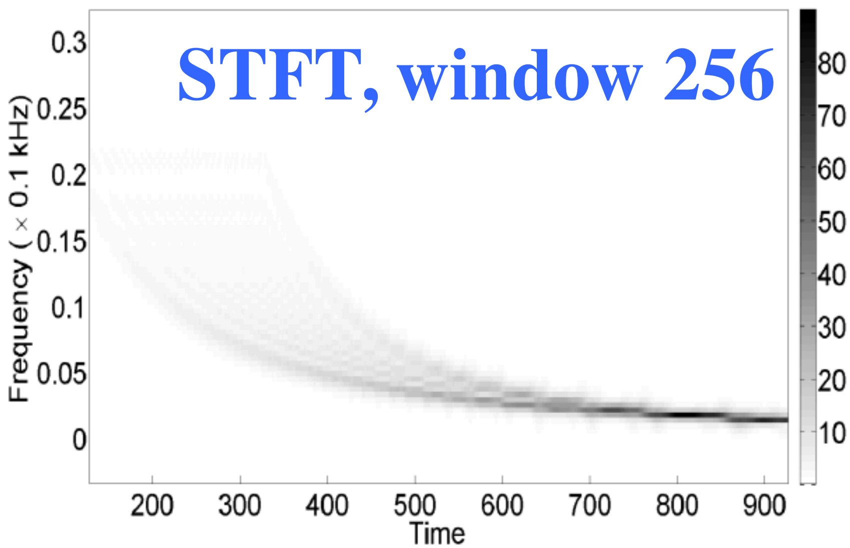
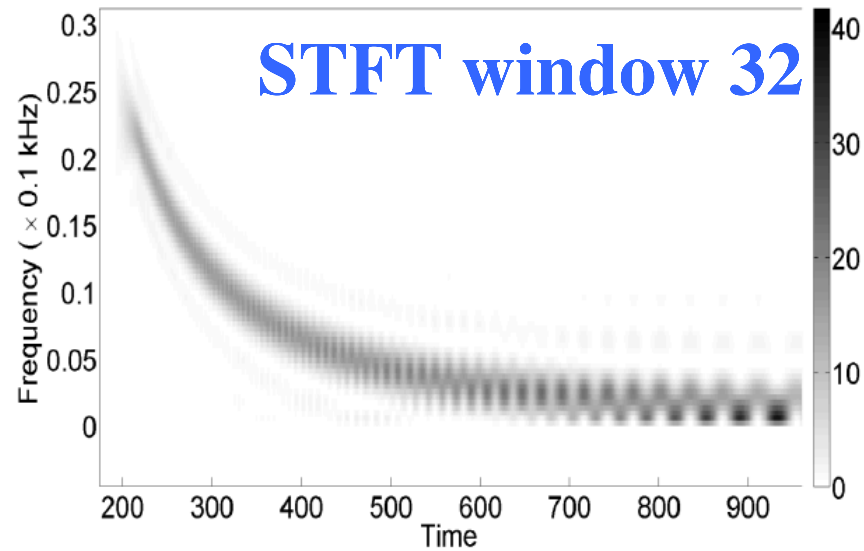
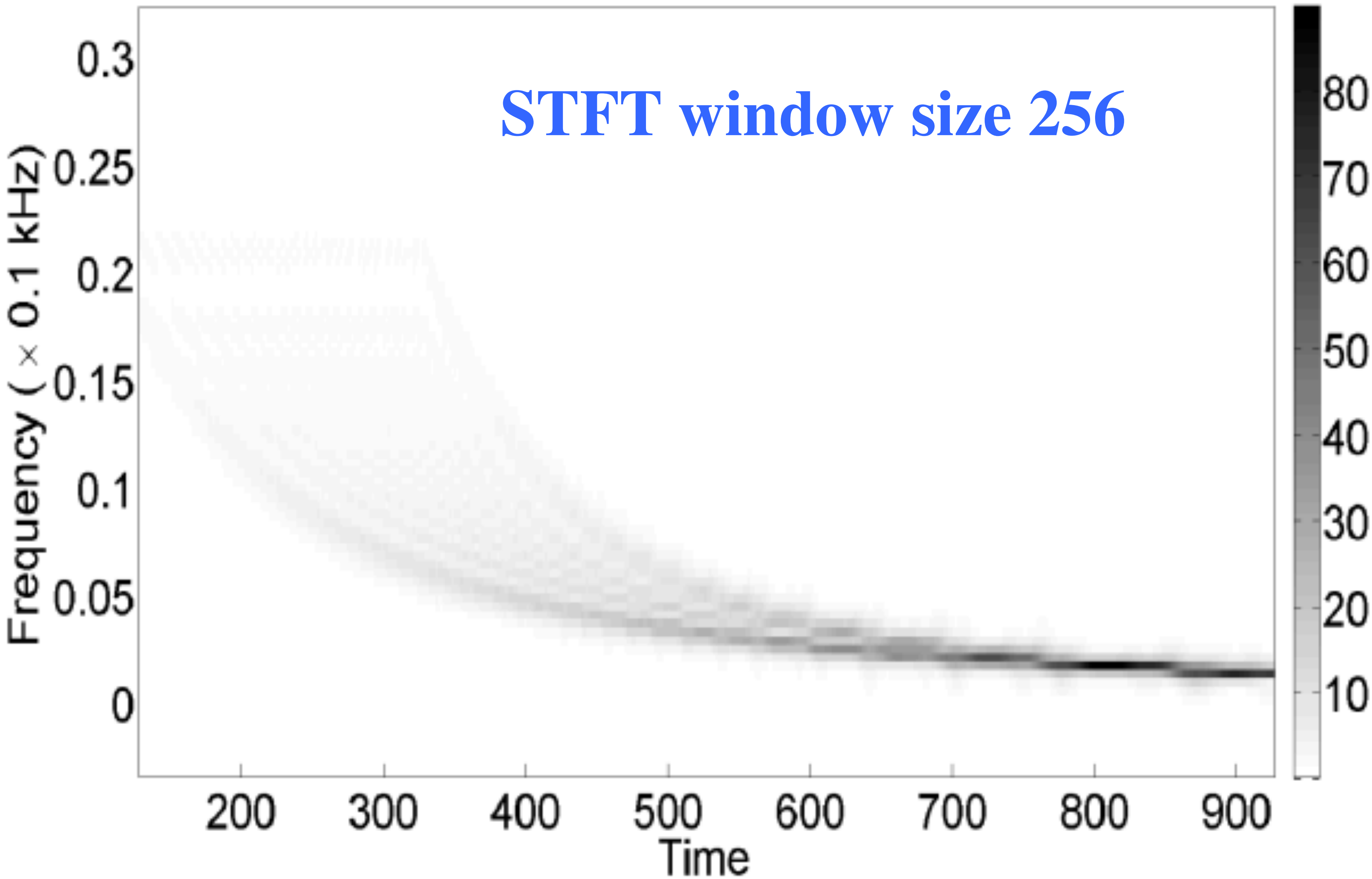
Ramanujan filter bank

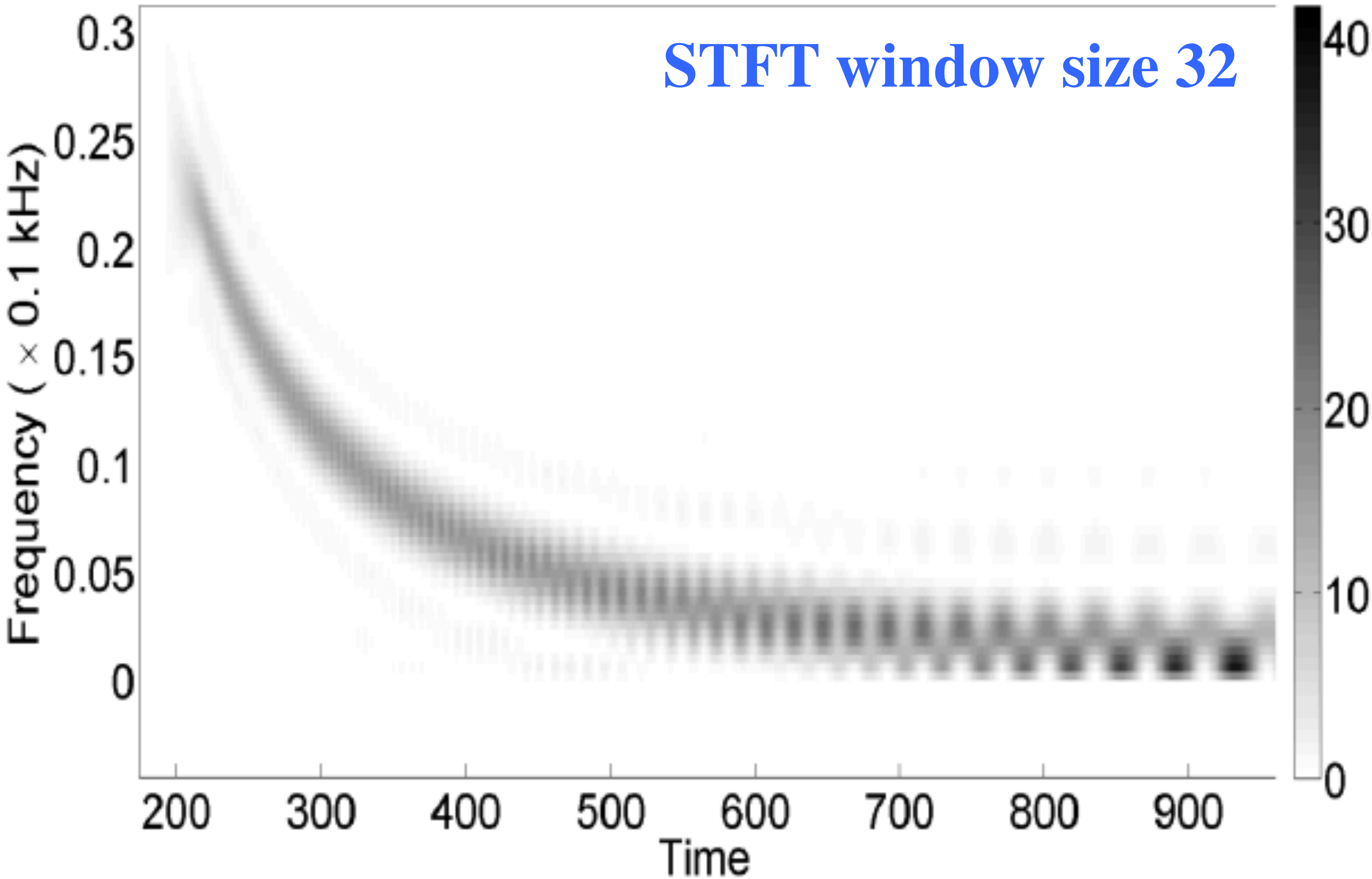


Fourier transform does not work





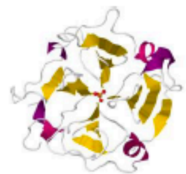
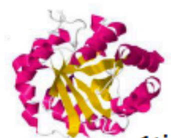

STFT window size 256
Frequency (× 0.1 kHz)
0.3
0.25
0.2
0.15
0.1
0.05
0
200
300
400
500
600
700
800
900
Time
80
70
60
50
40
30
20
10


STFT window size 32
Frequency (× 0.1 kHz)
0.3
0.25
0.2
0.15
0.1
0.05
0
200
300
400
500
600
700
800
900
Time
40
30
20
10
0

Protein repeats

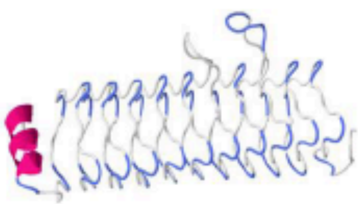
CS and Bio-Info methods

Ramanujan filter bank

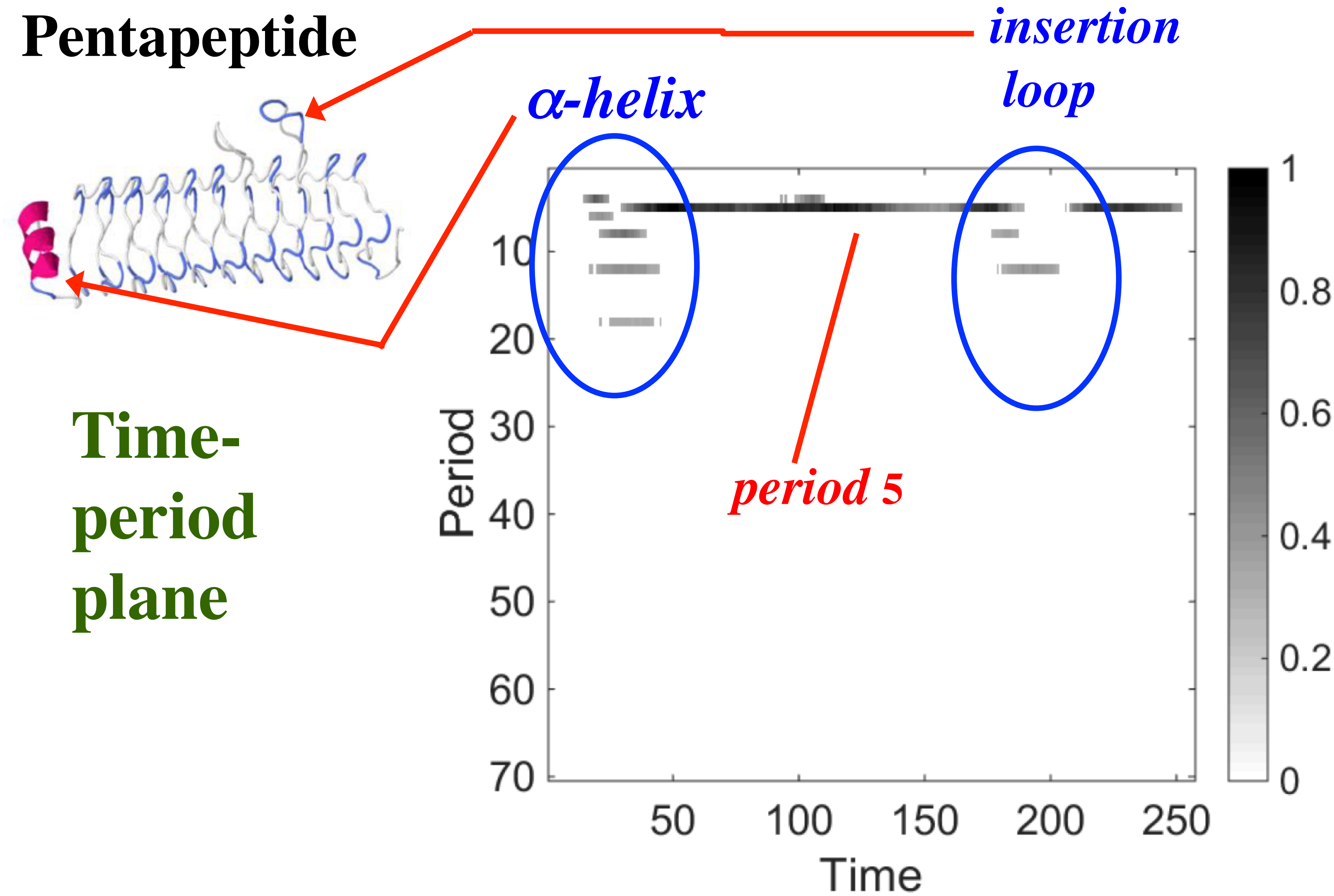
Repeat type	PDB ID	FTw.	WAV.	RAD.	REPw	RFB
β propeller  1hxn	1hxn	✗	✓	✓	✗	✓
TIM barrel  1tim (chain A)	1tim	✗	✓	✗	✗	✓
LRR  1lrv	1dfj	✗	✓	✓	✓	✓
	1lrv	✗	N.A.	✓	✓	✓
	4cil	✗	N.A.	✓	✓	✓
HEAT  1b3u	1b3u	✗	N.A.	✓	✓	✓

CS and Bio-Info methods

Ramanujan filter bank

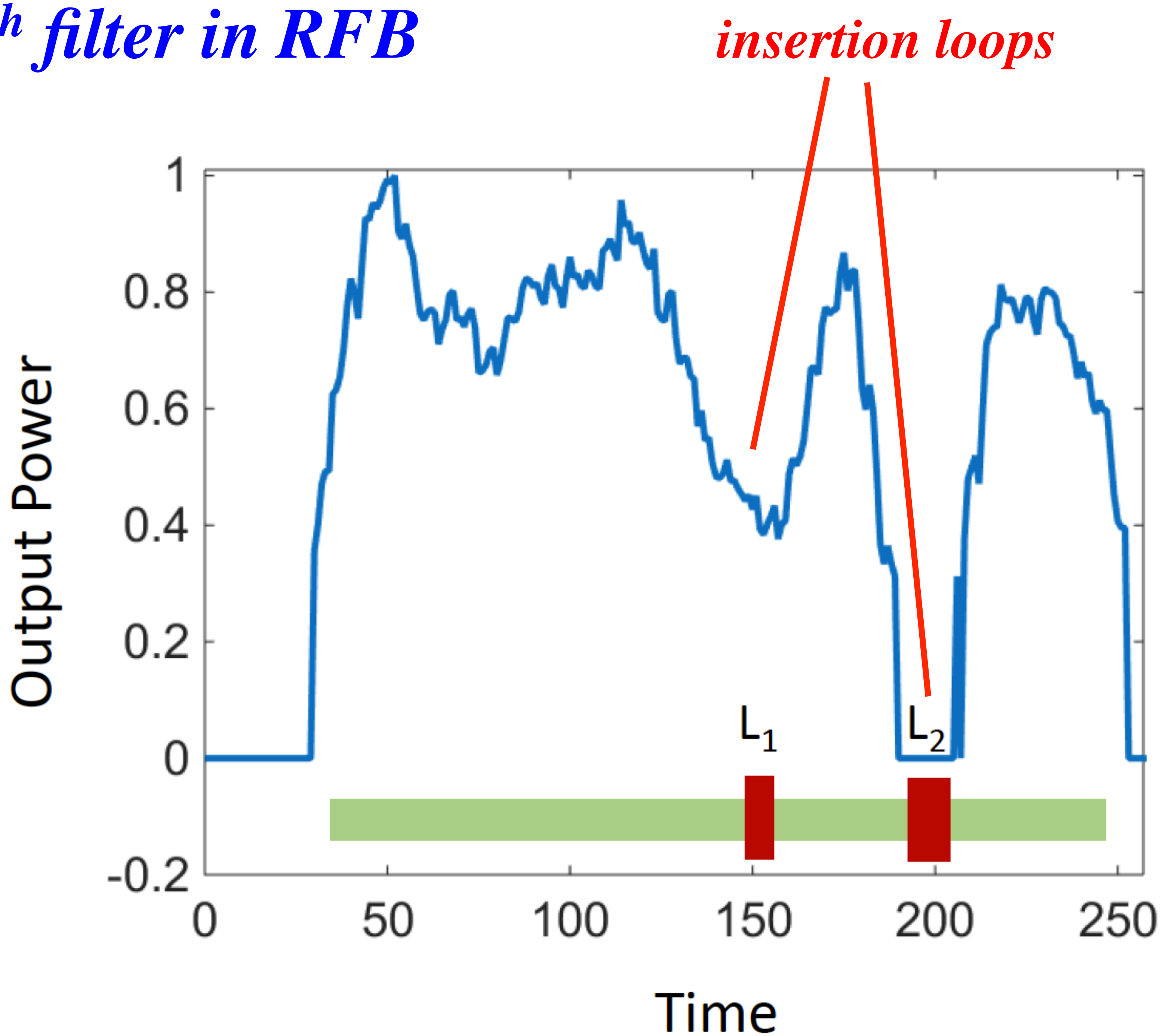
Ankyrin  1n11	1n11	✗	N.A.	✓	✓	✓
	GI: 30-425444	✗	N.A.	✓	✓	✓
Armadillo  3wpt	3wpt	✗	N.A.	✓	✓	✓
Pentapeptide  3du1	3du1	✗	N.A.	✗	✓	✓
	2bm4	✗	N.A.	✗	✓	✓
	3n90	✗	N.A.	✗	✓	✓

Pentapeptide



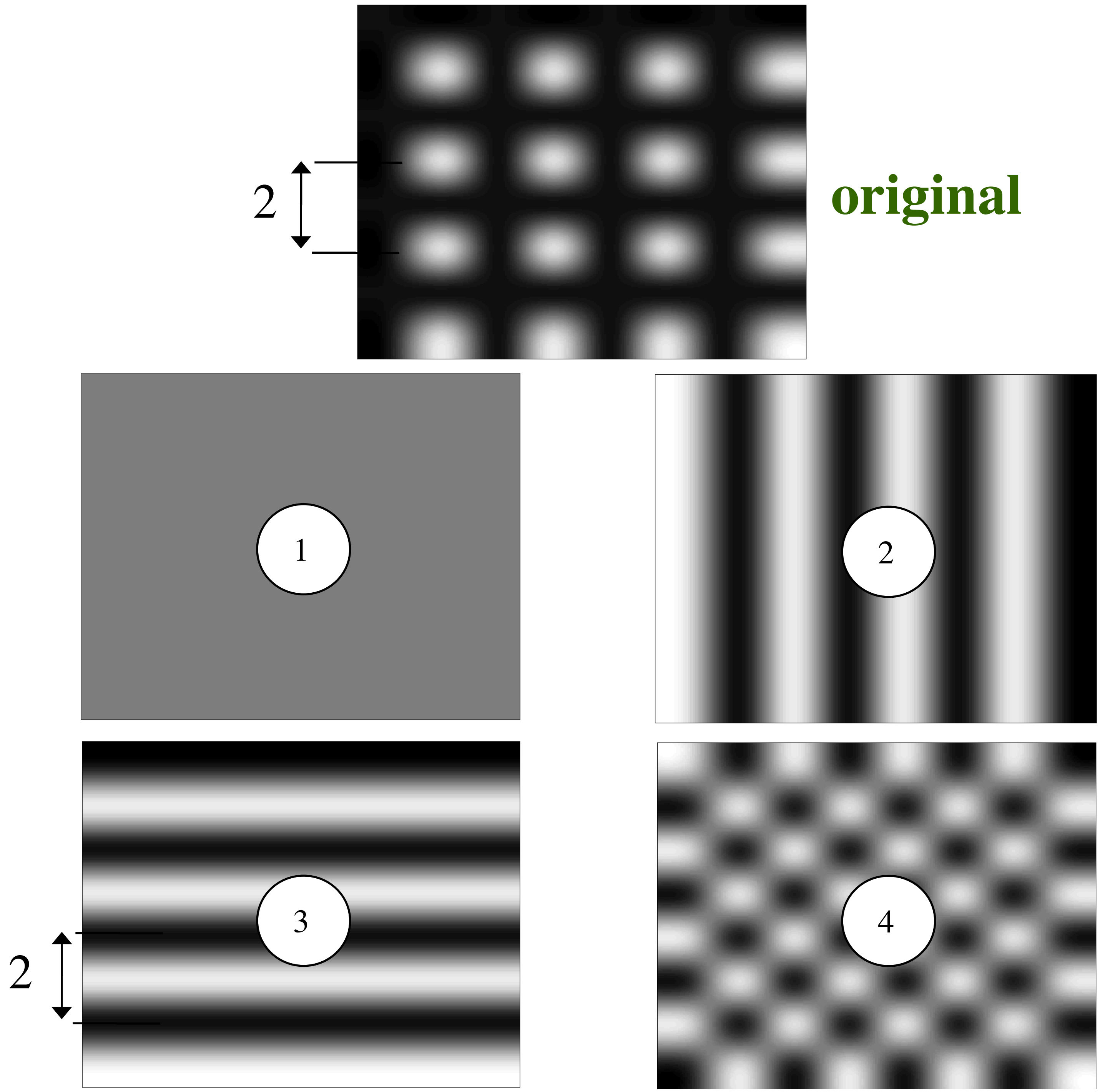
Time-period plane

5^{th} filter in RFB



2D Ramanujan space decompositions

ICASSP **2015**



2
original
1
2
3
4
2

History of Number theory in EE …

Coding in digital communications

Early DSP: NTT, Winograd …

Acoustics in buildings

Array processing

⋮

J. Maddox, "Möbius and problems of inversion," Nature, vol. 344, no. 29, p. 377, March 1990.

M. R. Schroeder, "The unreasonable effectiveness of number theory in science and communication," IEEE ASSP Magazine, pp. 5–12, Jan. 1988.

So, Hardy is wrong!

G. H. Hardy
1877 - 1947

The ‘real’ mathematics of the ‘real’ mathematicians is almost wholly ‘useless’.

The ‘real mathematician’ has his conscience clear.

Mathematics is a harmless and innocent occupation.

Applied mathematics, which is ‘useful’, is trivial.

Thank you!